

On linear independence measures of the values of Mahler functions

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Abstract

In this paper, we estimate the linear independence measures for the values of a class Mahler functions of degree one and two. For the purpose, we study the determinants of suitable Hermite-Padé approximation polynomials. Based on the non-vanishing of these determinants, we apply the functional equations to get an infinite sequence of approximations which is used to produce the linear independence measures.

1 Introduction and results

In the present work our aim is to obtain linear independence measures for the values of a class of Mahler functions $F(z), G(z) \in \mathbb{Q}[[z]]$ converging on some open disc $D_r := \{z : |z| < r \leq 1\}$ and satisfying a system of Mahler type functional equations

$$\begin{cases} F(z^d) = p_{11}(z)F(z) + p_{12}(z)G(z) + p_{10}(z), \\ G(z^d) = p_{21}(z)F(z) + p_{22}(z)G(z) + p_{20}(z) \end{cases} \quad (1)$$

with $p_{ij}(z) \in \mathbb{Q}(z)$ satisfying $p_{11}(z)p_{22}(z) - p_{12}(z)p_{21}(z) \neq 0$. Note that Mahler functions of degree one or two satisfy functional equations of the above type, if $F(z)$ and $G(z)$ are Mahler functions of degree one, then $p_{12}(z) = p_{21}(z) = 0$, and if $F(z)$ is of degree two, then we choose $G(z) = F(z^d)$. Our general result (Theorem 6 in Sec. 4) needs some technical notations to be presented later, and therefore to introduce our results we give here applications to some well-known functions.

The linear independence measures studied here are lower bounds for linear forms (in 1 and certain numbers γ_1 and γ_2) of the form

$$|h_0 + h_1\gamma_1 + h_2\gamma_2| > CH^{-\mu} \quad (2)$$

valid for any integers h_0, h_1, h_2 , not all zero, where the exponent μ is given explicitly, $H = \max\{|h_1|, |h_2|, H_0\}$, and positive constants C and H_0 are independent of h_i . In our results γ_1 and γ_2 are the values of the functions under consideration at rational points $a/b \in D_r \setminus \{0\}$, where $\log |a| = \lambda \log b$ ($0 \leq \lambda < \log(rb)/\log b$). We note that generally [12, Theorem 4.4.1] implies the existence of a μ (≥ 2) in our cases below, and here our aim is to obtain an explicit upper bound for the linear independence exponent

$$\mu(\gamma_1, \gamma_2) := \inf\{\mu : (2) \text{ holds for some } C > 0, H_0 > 0\}.$$

This work is a continuation to [14], where we studied simultaneous approximations of similar numbers γ_1 and γ_2 . We also note that, after Bugeaud's remarkable work [4] on Thue-Morse numbers, there has appeared several works on the irrationality exponents of the values of degree one Mahler functions, see [2, 5, 9, 11, 13, 15] and the references in [5]. In particular, the irrationality exponents of the numbers in Theorem 1-3 below equal 2.

1.1 Thue-Morse number and its square

Our first result studies the product

$$T(z) = \prod_{j=0}^{\infty} (1 - z^{2^j}),$$

the generating function of the Thue-Morse sequence on $\{-1, 1\}$, satisfying

$$T(z) = (1 - z)T(z^2). \quad (3)$$

Theorem 1. *We have*

$$\mu\left(T\left(\frac{1}{b}\right), T^2\left(\frac{1}{b}\right)\right) \leq \frac{91}{32} \approx 2.843 \dots$$

More generally, if $0 \leq \lambda < 7/29$, then

$$\mu\left(T\left(\frac{a}{b}\right), T^2\left(\frac{a}{b}\right)\right) \leq \frac{91}{32 - 104\lambda}.$$

1.2 Stern's sequence and its twisted version

Next, let $A(z)$ and $B(z)$ be generating functions of Stern's diatomic sequence and its twisted version. These functions satisfy functional equations

$$A(z) = (1 + z + z^2)A(z^2), \quad B(z) = 2 - (1 + z + z^2)B(z^2), \quad (4)$$

of type (I), see e.g. [7].

Theorem 2. *We have*

$$\mu\left(A\left(\frac{1}{b}\right), B\left(\frac{1}{b}\right)\right) \leq \frac{26}{9} \approx 2.888 \dots$$

More generally,

$$\mu\left(A\left(\frac{a}{b}\right), B\left(\frac{a}{b}\right)\right) \leq \begin{cases} \frac{130}{45 - 149\lambda}, & \text{if } \lambda < \frac{145}{1289}, \\ \frac{69}{25 - 89\lambda}, & \text{if } \frac{145}{1289} \leq \lambda < \frac{5}{29}. \end{cases}$$

1.3 Lambert series $G_3(z)$ and $F_3(z)$

The functions

$$G_3(z) = \sum_{j=0}^{\infty} \frac{z^{3^j}}{1 - z^{3^j}}, \quad F_3(z) = \sum_{j=0}^{\infty} \frac{z^{3^j}}{1 + z^{3^j}} = -G_3(-z)$$

satisfy

$$(1 - z)G_3(z) - (1 - z)G_3(z^3) - z = 0, \quad (1 + z)F_3(z) - (1 + z)F_3(z^3) - z = 0. \quad (5)$$

The following result studies the values of these typical examples of Mahler functions.

Theorem 3. *We have*

$$\mu\left(G_3\left(\frac{1}{b}\right), F_3\left(\frac{1}{b}\right)\right) \leq \frac{129}{37} \approx 3.486 \dots$$

More generally,

$$\mu\left(G_3\left(\frac{a}{b}\right), F_3\left(\frac{a}{b}\right)\right) \leq \begin{cases} \frac{129}{37 - 119\lambda}, & \text{if } \lambda < \frac{25}{443}, \\ \frac{83}{24 - 80\lambda}, & \text{if } \frac{25}{443} \leq \lambda < \frac{43}{337}, \\ \frac{57}{17 - 59\lambda}, & \text{if } \frac{43}{337} \leq \lambda < \frac{7}{29}. \end{cases}$$

1.4 The Rudin-Shapiro sequence

Let $(r_n)_{n \geq 0}$ be the Rudin-Shapiro sequence defined by $r_0 = 1, r_{2n} = r_n, r_{2n+1} = (-1)^n r_n$. Its generating function $R(z) = \sum_{n \geq 0} r_n z^n$ satisfies

$$R(z) = R(z^2) + zR(-z^2). \quad (6)$$

We shall investigate the values of $R(z)$ and $R(-z)$ at some rational points.

Theorem 4. *We have*

$$\mu\left(R\left(\frac{1}{b}\right), R\left(-\frac{1}{b}\right)\right) \leq \frac{13}{4} = 3.25.$$

More generally,

$$\mu\left(R\left(\frac{a}{b}\right), R\left(-\frac{a}{b}\right)\right) \leq \begin{cases} \frac{39}{12 - 40\lambda}, & \text{if } \lambda < \frac{21}{187}, \\ \frac{47}{15 - 53\lambda}, & \text{if } \frac{21}{187} \leq \lambda < \frac{3}{13}. \end{cases}$$

1.5 A degree 2 Mahler function

As an example of degree 2 Mahler functions we take the function $S(z)$ satisfying $S(0) = 1$ and

$$zS(z) - (1 + z + z^2)S(z^4) + S(z^{16}) = 0. \quad (7)$$

This function was introduced by Dilcher and Stolarsky [10], and it has been studied recently in several works, see e.g. [1], [3] and [8], in particular the algebraic independence of $S(\alpha)$, $S'(\alpha)$, $S(\alpha^4)$ and $S'(\alpha^4)$ is proved in [3] for all algebraic α , $0 < |\alpha| < 1$. Note also that in [8] an upper bounded 5 is obtained for the irrationality exponent of $S(1/b)$.

Theorem 5. *We have*

$$\mu\left(S\left(\frac{1}{b}\right), S\left(\frac{1}{b^4}\right)\right) \leq \frac{167}{25} = 6.68.$$

More generally, if $0 \leq \lambda < 1/5$, then

$$\mu\left(S\left(\frac{a}{b}\right), S\left(\left(\frac{a}{b}\right)^4\right)\right) \leq \frac{167}{25 - 93\lambda}.$$

All results above are based on non-vanishing of the determinants of suitable Hermite-Padé approximation polynomials studied in Section 2. This non-vanishing is verified here by computing the determinants, but it would be of great interest to find a more general criterion for this. After having some non-zero determinants the functional equations can be used to produce a sufficiently dense infinite sequence of approximations with non-zero determinants. It is well-known that such approximations can be used to produce linear independence measures. Section 3 contains this consideration, and it is then applied to prove a general result in Section 4. The proofs of Theorems 1-5 are given in Section 5.

2 Important determinants

We first note that the above system (1) can be given in the form

$$P(z)F(z^d) = P_{11}(z)F(z) + P_{12}(z)G(z) + P_{10}(z), \quad (8)$$

$$P(z)G(z^d) = P_{21}(z)F(z) + P_{22}(z)G(z) + P_{20}(z), \quad (9)$$

where $P(z)$, the least common denominator of $p_{ij}(z)$, and $P_{ij}(z) = P(z)p_{ij}(z)$ belong to $\mathbb{Z}[z]$ and satisfy $P_{11}(z)P_{22}(z) - P_{12}(z)P_{21}(z) \neq 0$.

For an integer $k \geq 1$, let $A_k(z), B_k(z), C_k(z) \in \mathbb{Z}[z]$ denote $(d_1, d_2, d_3) = (d_1(k), d_2(k), d_3(k))$ Hermite-Padé approximation polynomials of $F(z), G(z)$ and 1, so

$$A_k(z)F(z) + B_k(z)G(z) + C_k(z) = R_k(z), \quad (10)$$

where $\deg A_k(z) \leq d_1$, $\deg B_k(z) \leq d_2$, $\deg C_k(z) \leq d_3$ and the order of zero at $z = 0$ of the remainder term $R_k(z)$ satisfies $\text{ord } R_k(z) =: o(k) \geq d_1 + d_2 + d_3 + 2$. Clearly such polynomials, where at least one of $A_k(z), B_k(z)$ is not zero, exist. Substituting in (10) z^d for z and applying (8) and (9), we obtain

$$\begin{aligned} (P_{11}(z)A_k(z^d) + P_{21}(z)B_k(z^d))F(z) + (P_{12}(z)A_k(z^d) + P_{22}(z)B_k(z^d))G(z) \\ + P_{10}(z)A_k(z^d) + P_{20}(z)B_k(z^d) + P(z)C_k(z^d) = P(z)R_k(z^d). \end{aligned}$$

Repeating this procedure m times, we have

$$A_{k,m}(z)F(z) + B_{k,m}(z)G(z) + C_{k,m}(z) = R_{k,m}(z), \quad m = 0, 1, \dots, \quad (11)$$

where $A_{k,0}(z) = A_k(z)$, $B_{k,0}(z) = B_k(z)$, $C_{k,0}(z) = C_k(z)$, $R_{k,0}(z) = R_k(z)$ and, for $m = 1, 2, \dots$,

$$\begin{cases} A_{k,m}(z) = P_{11}(z)A_{k,m-1}(z^d) + P_{21}(z)B_{k,m-1}(z^d), \\ B_{k,m}(z) = P_{12}(z)A_{k,m-1}(z^d) + P_{22}(z)B_{k,m-1}(z^d), \\ C_{k,m}(z) = P_{10}(z)A_{k,m-1}(z^d) + P_{20}(z)B_{k,m-1}(z^d) \\ \quad + P(z)C_{k,m-1}(z^d), \\ R_{k,m}(z) = P(z)R_{k,m-1}(z^d). \end{cases} \quad (12)$$

We are interested in determinants

$$\Delta(\underline{k}, m, z) := \det \begin{pmatrix} A_{k_1,m}(z) & B_{k_1,m}(z) & C_{k_1,m}(z) \\ A_{k_2,m}(z) & B_{k_2,m}(z) & C_{k_2,m}(z) \\ A_{k_3,m}(z) & B_{k_3,m}(z) & C_{k_3,m}(z) \end{pmatrix},$$

where $1 \leq k_1 < k_2 < k_3$. By the above recursions (12)

$$\Delta(\underline{k}, m, z) = \Phi(z)\Delta(\underline{k}, m-1, z^d), \quad \Phi(z) := (P_{11}(z)P_{22}(z) - P_{12}(z)P_{21}(z))P(z),$$

and so

$$\Delta(\underline{k}, m, z) = \Delta(\underline{k}, 0, z^{d^m}) \prod_{j=0}^{m-1} \Phi(z^{d^j}). \quad (13)$$

In particular, for degree one functions we have $\Phi(z) = P_{11}(z)P_{22}(z)P(z)$, and for degree two function $F(z)$ with $G(z) = F(z^d)$ the function $\Phi(z) = P_{21}(z)P^2(z)$.

Let $\bar{d}(k) := \max\{d_1(k), d_2(k), d_3(k)\}$. By our assumption $k_1 < k_2 < k_3$ it is natural to assume that $\bar{d}(k_1) \leq \bar{d}(k_2) \leq \bar{d}(k_3)$ and $o(k_1) \leq o(k_2) \leq o(k_3)$. Since

$$\Delta(\underline{k}, 0, z) = \det \begin{pmatrix} A_{k_1}(z) & B_{k_1}(z) & R_{k_1}(z) \\ A_{k_2}(z) & B_{k_2}(z) & R_{k_2}(z) \\ A_{k_3}(z) & B_{k_3}(z) & R_{k_3}(z) \end{pmatrix},$$

it follows that $o(k_1) \leq \text{ord } \Delta(\underline{k}, 0, z) \leq \deg \Delta(\underline{k}, 0, z) \leq \bar{d}(k_1) + \bar{d}(k_2) + \bar{d}(k_3)$, if $\Delta(\underline{k}, 0, z) \neq 0$. Thus in this case

$$\Delta(\underline{k}, 0, z) =: z^{o(k_1)} D(\underline{k}, z) \quad (14)$$

with some polynomial $D(\underline{k}, z) \neq 0$, $\deg D(\underline{k}, z) \leq \bar{d}(k_1) + \bar{d}(k_2) + \bar{d}(k_3) - o(k_1)$. Further, if $o(k_1) > \bar{d}(k_1) + \bar{d}(k_2) + \bar{d}(k_3)$, then $\Delta(\underline{k}, 0, z) = 0$.

We note that the condition $D(\underline{k}, z) \neq 0$ gives a strong restriction to $o(k_1)$. For example, if $d_j(k_1) = k$, $d_j(k_2) = k + 1$, $d_j(k_3) = k + 2$ ($j = 1, 2, 3$), then $\deg \Delta(\underline{k}, 0, z) \leq 3k + 3$ and $o(k_1) \geq 3k + 2$. Thus the condition $D(\underline{k}, z) \neq 0$ is possible only if $3k + 2 \leq o(k_1) \leq 3k + 3$.

The above means that one determinant $\Delta(\underline{k}, 0, z) \neq 0$ gives an infinite sequence of determinants $\Delta(\underline{k}, m, z) \neq 0$, $m = 0, 1, \dots$. When considering the values of the functions at rational points $z = a/b$ we need to know that $\Delta(\underline{k}, m, a/b) \neq 0$ at least for all sufficiently large m . This condition can be verified in many concrete cases by using (13) and (14), since $\deg D(\underline{k}, z)$ is small.

3 Fundamental lemma

In this section γ_1 and γ_2 denote real numbers and $b \geq 2$ is an integer. Let $\underline{k} = \underline{k}(\ell) = (k_{\ell,1}, k_{\ell,2}, k_{\ell,3})$ ($\ell = 1, \dots, L$) be vectors with positive integer components $k_{\ell,i}$ satisfying $k_{\ell,1} < k_{\ell,2} < k_{\ell,3}$ and $k_{\ell,3} \leq k_{\ell+1,1}$ ($\ell = 1, \dots, L-1$), $k_{L,3} \leq dk_{1,1}$. Assume that for each $k = k_{\ell,i}$ there exists an integer $m_0(k)$ such that for all $m \geq m_0(k)$ we have linear forms

$$a_{k,m}\gamma_1 + b_{k,m}\gamma_2 + c_{k,m} = r_{k,m}$$

with the following properties (i) – (iii).

(i) The coefficients $a_{k,m}, b_{k,m}, c_{k,m} \in \mathbb{Z}$ and satisfy

$$\max\{|a_{k,m}|, |b_{k,m}|\} \leq c_1(k)b^{E(k)d^m}, \quad (15)$$

where $E(k)$ and $c_1(k)$ (as also $c_2(k), \dots$ later) are positive constants independent of m .

(ii) We have

$$|r_{k,m}| \leq c_2(k)b^{-V(k)d^m}, \quad (16)$$

where $V(k) > 0$ is independent of m .

(iii) The determinant

$$\det \begin{pmatrix} a_{k_{\ell,1},m} & b_{k_{\ell,1},m} & c_{k_{\ell,1},m} \\ a_{k_{\ell,2},m} & b_{k_{\ell,2},m} & c_{k_{\ell,2},m} \\ a_{k_{\ell,3},m} & b_{k_{\ell,3},m} & c_{k_{\ell,3},m} \end{pmatrix} \neq 0.$$

for all $\ell = 1, \dots, L; m \geq m_0(\underline{k}(\ell)) = \max_{1 \leq i \leq 3} \{m_0(k_{\ell,i})\}$.

For the following fundamental lemma, we finally define, for all $\ell = 1, \dots, L$, the notations

$$\begin{aligned}\theta(\ell) &= \max_{1 \leq i < j \leq 3} \{E(k_{\ell,i}) + E(k_{\ell,j})\}, \\ \nu(\ell) &= \min_{\substack{1 \leq i, j \leq 3 \\ i \neq j}} \{V(k_{\ell,i}) - E(k_{\ell,j})\},\end{aligned}$$

and denote $K := (\underline{k}(1), \dots, \underline{k}(L))$.

Lemma 1. *Suppose that $0 < \nu(1) < \dots < \nu(L) < d\nu(1)$. Then there exist positive constants $C = C(K)$ and $H_0 = H_0(K)$ such that for any integers h_0, h_1, h_2 , not all zero,*

$$|h_0 + h_1\gamma_1 + h_2\gamma_2| > CH^{-\mu},$$

where $H = \max\{|h_1|, |h_2|, H_0\}$ and

$$\mu = \max_{1 \leq \ell \leq L} \mu(\ell), \quad \mu(\ell) := \frac{\theta(\ell+1)}{\nu(\ell)}, \quad \theta(L+1) := d\theta(1).$$

Proof. Let

$$\Lambda = h_0 + h_1\gamma_1 + h_2\gamma_2.$$

By the condition (iii) above, for all $\ell = 1, \dots, L$ there exist $1 \leq i < j \leq 3$ such that

$$D(\underline{k}(\ell), \underline{h}) := \det \begin{pmatrix} h_1 & h_2 & h_0 \\ a_{k_{\ell,i},m} & b_{k_{\ell,i},m} & c_{k_{\ell,i},m} \\ a_{k_{\ell,j},m} & b_{k_{\ell,j},m} & c_{k_{\ell,j},m} \end{pmatrix} = \det \begin{pmatrix} h_1 & h_2 & \Lambda \\ a_{k_{\ell,i},m} & b_{k_{\ell,i},m} & r_{k_{\ell,i},m} \\ a_{k_{\ell,j},m} & b_{k_{\ell,j},m} & r_{k_{\ell,j},m} \end{pmatrix} \neq 0.$$

Since $D(\underline{k}(\ell), \underline{h})$ is an integer, we obtain, by (15) and (16),

$$\begin{aligned}1 \leq 2|\Lambda| c_1(k_{\ell,i})c_1(k_{\ell,j})b^{(E(k_{\ell,i})+E(k_{\ell,j}))d^m} + 2hc_1(k_{\ell,j})c_2(k_{\ell,i})b^{-(V(k_{\ell,i})-E(k_{\ell,j}))d^m} + \\ 2hc_1(k_{\ell,i})c_2(k_{\ell,j})b^{-(V(k_{\ell,j})-E(k_{\ell,i}))d^m}\end{aligned} \quad (17)$$

with $h = \max\{|h_1|, |h_2|\}$. The definitions of $\theta(\ell)$ and $\nu(\ell)$ then give

$$1 \leq C_1(K) |\Lambda| b^{\theta(\ell)d^m} + C_2(K) hb^{-\nu(\ell)d^m} \quad (18)$$

for all $m \geq M_0 := \max\{m_0(\underline{k}(1)), \dots, m_0(\underline{k}(L))\}$, and here $C_1(K)$ and $C_2(K)$ (and also $C_3(K)$ later) are positive constants depending on K . Note that $C_1(K)$ and $C_2(K)$ are the same for all ℓ .

We now choose H_0 in such a way that

$$2C_2(K)H_0 \geq b^{\nu(1)d^{M_0}},$$

and fix the pair (ℓ, m) from the sequence $(1, M_0), \dots, (L, M_0), (1, M_0 + 1), \dots, (L, M_0 + 1), (1, M_0 + 2), \dots$ to be the first one satisfying

$$2C_2(K)H < b^{\nu(\ell)d^m},$$

where $H = \max\{h, H_0\}$. Then $(\ell, m) \neq (1, M_0)$, and the pair just before it is $(\ell - 1, m)$, if $\ell > 1$, and $(L, m - 1)$, if $\ell = 1$. The above choice means that

$$2C_2(K)H \geq b^{\nu(\ell-1)d^m}, \quad \ell > 1,$$

$$2C_2(K)H \geq b^{\nu(L)d^{m-1}}, \quad \ell = 1.$$

In the first case, by (18),

$$\frac{1}{2} < C_1(K) |\Lambda| b^{\theta(\ell)d^m} = C_1(K) |\Lambda| (b^{\nu(\ell-1)d^m})^{\theta(\ell)/\nu(\ell-1)} \leq C_3(K) |\Lambda| H^\mu.$$

In the case $\ell = 1$ we similarly have

$$\frac{1}{2} < C_1(K) |\Lambda| b^{\theta(1)d^m} = C_1(K) |\Lambda| (b^{\nu(L)d^{m-1}})^{d\theta(1)/\nu(L)} \leq C_3(K) |\Lambda| H^\mu.$$

This proves our lemma. \square

4 General theorem

We now assume that $F(z), G(z) \in \mathbb{Q}[[z]]$ converge in some disk D_r and satisfy (8) and (9). Our aim is to apply Lemma 1 to consider the function values $F(a/b)$ and $G(a/b)$ at non-zero rational points $a/b \in D_r$, where $\log |a| = \lambda \log b$, $0 \leq \lambda < \log(rb)/\log b$. We also assume that

$$(P_{11}((a/b)^{d^j})P_{22}((a/b)^{d^j}) - P_{12}((a/b)^{d^j})P_{21}((a/b)^{d^j}))P((a/b)^{d^j}) \neq 0, \quad j = 0, 1, \dots \quad (19)$$

The approximation forms we use are obtained from (11) at $z = a/b$. The recursions (12) imply, for all $m \geq 1$,

$$\deg A_{k,m}(z), \deg B_{k,m}(z), \deg C_{k,m}(z) \leq \left(\bar{e}(k) + \frac{\tau}{d-1} \right) \cdot d^m - \frac{\tau}{d-1} \quad (20)$$

where $\bar{e}(k)$ and τ are non-negative integers satisfying $\bar{e}(k) \leq \bar{d}(k) := \max\{d_1(k), d_2(k), d_3(k)\}$ and $\tau \leq \nu$, the maximum of the degrees of $P_{ij}(z)$ and $P(z)$. Thus the multiplication of (11) at $z = a/b$ by

$$Q_{k,m} := b^{(\bar{e}(k) + \frac{\tau}{d-1})d^m - \frac{\tau}{d-1}}$$

leads to linear forms

$$a_{k,m}F\left(\frac{a}{b}\right) + b_{k,m}G\left(\frac{a}{b}\right) + c_{k,m} = r_{k,m}, \quad m = 0, 1, \dots,$$

where all $a_{k,m}, b_{k,m}$ and $c_{k,m}$ are integers. To be able to apply Lemma 1 with $\gamma_1 = F(a/b), \gamma_2 = G(a/b)$ we need to estimate the coefficients $a_{k,m}$ and $b_{k,m}$ and the remainders $r_{k,m}$. For this we apply the recursions (12).

Let $\tilde{P}(z)$ denote the polynomial, where the coefficient of z^j is the maximum of the absolute values of the corresponding coefficients in $P_{ij}(z), 1 \leq i, j \leq 2$. Then, for all $m = 1, 2, \dots$,

$$\begin{aligned} |A_{k,m}(z)| &\leq \tilde{P}(|z|) \left(|A_{k,m-1}(z^d)| + \delta |B_{k,m-1}(z^d)| \right), \\ |B_{k,m}(z)| &\leq \tilde{P}(|z|) \left(\delta |A_{k,m-1}(z^d)| + |B_{k,m-1}(z^d)| \right), \end{aligned}$$

where $\delta = 0$ for degree one functions $F(z)$ and $G(z)$, and $\delta = 1$ otherwise. Applying these inequalities we obtain

$$\max\{|A_{k,m}(z)|, |B_{k,m}(z)|\} \leq (1 + \delta)^m \max\{|A_k(z^{d^m})|, |B_k(z^{d^m})|\} \prod_{j=0}^{m-1} \tilde{P}(|z|^{d^j}).$$

Therefore, for all $m \geq m_1(k)$,

$$\max\{|a_{k,m}|, |b_{k,m}|\} \leq c_3(k) b^{(\bar{e}(k) + \frac{\tau}{d-1})d^m},$$

if the condition

$$(1 + \delta) \left| \tilde{P}(0) \right| \leq 1 \quad (21)$$

holds. Generally, for any given $\delta_1 > 0$,

$$\max\{|a_{k,m}|, |b_{k,m}|\} \leq c_3(k) b^{(\bar{e}(k) + \frac{\tau}{d-1} + \delta_1)d^m} \quad (22)$$

for all $m \geq m_2(k, \delta_1)$, and under the condition (21) we may choose here $\delta_1 = 0$.

Since

$$R_{k,m}(z) = R_k(z^{d^m}) \prod_{j=0}^{m-1} P(z^{d^j}),$$

we also have

$$|r_{k,m}| \leq c_4(k) \max\{1, |P(0)|^m\} b^{-((1-\lambda)o(k) - \bar{e}(k) - \frac{\tau}{d-1})d^m}$$

for all $m \geq m_3(k)$. Thus, for any given $\delta_2 > 0$,

$$|r_{k,m}| \leq c_4(k) b^{-((1-\lambda)o(k) - \bar{e}(k) - \frac{\tau}{d-1} - \delta_2)d^m} \quad (23)$$

for all $m \geq m_4(k, \delta_2)$, and we may use here the value $\delta_2 = 0$, if the condition

$$|P(0)| \leq 1 \quad (24)$$

holds.

Thus we have the estimates (15) and (16) for all $m \geq m_5(k, \delta_1, \delta_2)$, where

$$E(k) = \bar{e}(k) + \frac{\tau}{d-1} + \delta_1, \quad V(k) = (1-\lambda)o(k) - \bar{e}(k) - \frac{\tau}{d-1} - \delta_2. \quad (25)$$

By using these values with Lemma 1 we get the following theorem, we only need to note that the condition $D(\underline{k}, z) \neq 0$ implies $D(\underline{k}, (a/b)^{d^m}) \neq 0$ for all $m \geq m_6(\underline{k}, a/b)$.

Theorem 6. Assume that the condition (19) holds and $D(\underline{k}, z) \neq 0$ for all $\ell = 1, \dots, L$. Let $\theta(\ell)$ and $\nu(\ell)$ be defined as in Lemma 1 with $E(k)$ and $V(k)$ given in (25). If $0 < \nu(1) < \dots < \nu(L) < d\nu(1)$, then there exist positive constants $\lambda_0 = \lambda_0(K, F, G)$, $C = C(K, a/b, F, G)$ and $H_0 = H_0(K, a/b, F, G)$ such that for all $0 \leq \lambda < \lambda_0$ and any integers h_0, h_1, h_2 , not all zero,

$$\left| h_0 + h_1 F\left(\frac{a}{b}\right) + h_2 G\left(\frac{a}{b}\right) \right| > CH^{-\mu}$$

with H and μ as in Lemma 1.

5 Proof of Theorems 1-5

We are ready to prove Theorems 1-5. We start by giving the following formulae which follow from (25):

$$\begin{cases} \theta(\ell) = \max_{1 \leq i < j \leq 3} \{\bar{e}(k_{\ell,i}) + \bar{e}(k_{\ell,j})\} + \frac{2\tau}{d-1} + 2\delta_1, \\ \nu(\ell) = \min_{\substack{1 \leq i, j \leq 3 \\ i \neq j}} \{(1-\lambda)o(k_{\ell,i}) - \bar{e}(k_{\ell,i}) - \bar{e}(k_{\ell,j})\} - \frac{2\tau}{d-1} - \delta_1 - \delta_2. \end{cases} \quad (26)$$

Thus we should choose τ and δ_i as small as possible while applying Lemma 1.

Proof of Theorem 1. To prove Theorem 1, we apply Theorem 6 with $F(z) = T(z)$, $G(z) = T^2(z)$. Now, by (3),

$$(1-z)^2 F(z^2) = (1-z)F(z), \quad (1-z)^2 G(z^2) = G(z).$$

Therefore $r = 1$, $\delta = 0$, $P(z) = (1-z)^2$, $\tilde{P}(z) = 1+z$ and $\tilde{P}(0) = P(0) = 1$ gives $\delta_1 = \delta_2 = 0$. We shall use $(k, k+1, k-1)$ approximations and we may take $\bar{e}(k) = k+1$, $\tau = 0$. Our $\underline{k}(\ell)$ are $(k_{\ell,1}, k_{\ell,1}+1, k_{\ell,1}+2)$ and the choices for $k = k_{\ell,1}$ are 29, 31, 34, 43 and 49. For all these values $o(k) = 3k+2$. Since $\deg \Delta(\underline{k}(\ell), z) \leq 3k+3$, we have $D(\underline{k}(\ell), z) = s_{\ell,0} + s_{\ell,1}z$, where

$$s_{\ell,0} = \det \begin{pmatrix} A_{k+1}(0) & B_{k+1}(0) \\ A_{k+2}(0) & B_{k+2}(0) \end{pmatrix} c \neq 0$$

where c is the coefficient of z^{3k+2} in $R_k(z)$ (see Appendix). In fact $s_{\ell,0}$ is nonzero in all of our cases, also in the proofs of Theorems 2-5. By using (26) we get

$$\theta(\ell) = 2k+5, \quad \nu(\ell) = k-2-\lambda(3k+2)$$

for all $\lambda < 2/3$. So we have the following table.

ℓ	1	2	3	4	5
k	29	31	34	43	49
$\theta(\ell)$	63	67	73	91	103
$\nu(\ell)$	$27-89\lambda$	$29-95\lambda$	$32-104\lambda$	$41-131\lambda$	$47-149\lambda$

For the condition $0 < \nu(1) < \dots < \nu(5) < 2\nu(1)$ we need to assume $\lambda < \lambda_0 := 7/29 \approx 0.241\dots$. When $\lambda < \lambda_0$, the comparison of $\mu(\ell)$ gives

$$\mu = \max_{\ell} \frac{\theta(\ell+1)}{\nu(\ell)} = \frac{\theta(4)}{\nu(3)} = \frac{91}{32-104\lambda}.$$

This prove Theorem 1. □

To prove Theorem 2-5, we need to modify the choices of parameters.

Proof of Theorem 2. Here we apply Theorem 6 with $F(z) = A(z)$, $G(z) = B(z)$, and the use of (4) gives $r = 1$, $\delta = 0$, $P(z) = 1+z+z^2$, $\tilde{P}(z) = 1$ and $\delta_1 = \delta_2 = 0$. The $(k, k+1, k-1)$ approximations give $\bar{e}(k) = k+1$, $\tau = 1$. By choosing $\underline{k}(\ell)$ as above, where $k = k_{\ell,1}$ are 29, 31, 34, 38, 43 and 49, we get $o(k) = 3k+2$ and the determinants $D(\underline{k}(\ell), z) \neq 0$ (see Appendix). Further,

$$\theta(\ell) = 2k+7, \quad \nu(\ell) = k-4-\lambda(3k+2)$$

for all $\lambda < 2/3$, and this leads to the following table.

ℓ	1	2	3	4	5	6
k	29	31	34	38	43	49
$\theta(\ell)$	65	69	75	83	93	105
$\nu(\ell)$	$25-89\lambda$	$27-95\lambda$	$30-104\lambda$	$34-116\lambda$	$39-131\lambda$	$45-149\lambda$

To satisfy the condition $0 < \nu(1) < \dots < \nu(6) < 2\nu(1)$, we need to assume $\lambda < \lambda_0 := 5/29 \approx 0.172\dots$. After the comparison of $\mu(\ell) = \theta(\ell+1)/\nu(\ell)$ we see that

$$\mu = \max_{1 \leq \ell \leq 6} \mu(\ell) = \begin{cases} \mu(6) = \frac{130}{45-149\lambda}, & \text{if } \lambda < \frac{145}{1289}, \\ \mu(1) = \frac{69}{25-89\lambda}, & \text{if } \frac{145}{1289} \leq \lambda < \frac{5}{29}. \end{cases}$$

This proves Theorem 2. □

Remark 1. We note that here all determinants $D(\underline{k}(\ell), z) \neq 0$, $1 \leq k \leq 50$. In all other theorems most of these determinants equal zero.

Proof of Theorem 3. In this case we apply Theorem 6 with $d = 3$, $F(z) = G_3(z)$ and $G(z) = F_3(z)$. Then (5) implies $r = 1$, $\delta = 0$, $P(z) = 1 - z^2$, $\tilde{P}(z) = 1 + z^2$ and $\delta_1 = \delta_2 = 0$. The use of (k, k, k) approximations give $\bar{e}(k) = k$, $\tau = 2$. If $\underline{k}(\ell)$ is the same as above and $k = k_{\ell,1}$ are 19, 26 and 39, then $o(k) = 3k + 2$ and $D(\underline{k}(\ell), z) \neq 0$ (see Appendix). By (26), if $\lambda < 2/3$, we get

$$\theta(\ell) = 2k + 5, \quad \nu(\ell) = k - 2 - \lambda(3k + 2).$$

Now the table is the following.

ℓ	1	2	3
k	19	26	39
$\theta(\ell)$	43	57	83
$\nu(\ell)$	$17 - 59\lambda$	$24 - 80\lambda$	$37 - 119\lambda$

The condition $0 < \nu(1) < \nu(2) < \nu(3) < 3\nu(1)$ holds, if $\lambda < \lambda_0 := 7/29 \approx 0.241 \dots$. Similarly to the above proofs we now get Theorem 3. \square

Proof of Theorem 4. Here we may use Theorem 6 with $F(z) = R(z)$ and $G(z) = R(-z)$. By (6), we have

$$2zF(z^2) = zF(z) + zG(z), \quad 2zG(z^2) = F(z) + G(z).$$

Therefore, we can choose $r = 1$, $\delta = 1$, $P(z) = 2z$, and $\tilde{P}(z) = 1 + z$. Since $P(0) = 0$, (23) holds and we may take $\delta_2 = 0$. We use the (k, k, k) approximations and we can take $\bar{e}(k) = k$ and $\tau = 1$. We also choose $\underline{k}(\ell) = (k_{\ell,1}, k_{\ell,1} + 1, k_{\ell,1} + 2)$ where $k = k_{\ell,1}$ are 17, 21 and 26. Then we get $o(k) = 3k + 2$ and the determinants $D(\underline{k}(\ell), z) \neq 0$ (see Appendix). Moreover,

$$\theta(\ell) = 2k + 5 + 2\delta_1, \quad \nu(\ell) = k - 2 - \delta_1 - \lambda(3k + 2)$$

for all $\lambda < 2/3$. This gives the following table.

ℓ	1	2	3
k	17	21	26
$\theta(\ell)$	$39 + 2\delta_1$	$47 + 2\delta_1$	$57 + 2\delta_1$
$\nu(\ell)$	$15 - \delta_1 - 53\lambda$	$19 - \delta_1 - 65\lambda$	$24 - \delta_1 - 80\lambda$

The condition $0 < \nu(1) < \nu(2) < \nu(3) < 2\nu(1)$ holds, if $\lambda < 3/13 \approx 0.230 \dots$ and δ_1 is sufficiently small. If $\lambda < 21/187 \approx 0.112 \dots$ and δ_1 is small enough, then

$$\mu = \frac{\theta(4)}{\nu(3)} = \frac{78 + 4\delta_1}{24 - \delta_1 - 80\lambda}.$$

If $21/187 \leq \lambda < 3/13$, then

$$\mu = \frac{\theta(2)}{\nu(1)} = \frac{47 + 2\delta_1}{15 - \delta_1 - 53\lambda}.$$

This proves Theorem 4, since we may choose δ_1 arbitrarily small. \square

Proof of Theorem 5. We now apply Theorem 6 with $F(z) = S(z)$ and $G(z) = S(z^4)$. The use of (7) gives $d = 4$, $r = 1$ and

$$F(z^4) = G(z), \quad G(z^4) = -zF(z) + (1 + z + z^2)G(z).$$

Since $P(0) = 1$, we may choose $\delta_2 = 0$ in (23). We shall use of $(k, k-1, k)$ approximations and we may take $\bar{e}(k) = k$, $\tau = 1$. Again our $\underline{k}(\ell) = (k_{\ell,1}, k_{\ell,1}+1, k_{\ell,1}+2)$ and the choices for $k = k_{\ell,1}$ are 10 and 26. For both of these values $o(k) = 3k+1$ and the determinants $D(\underline{k}(\ell), z) \neq 0$ (see Appendix). By using (26), if $\lambda < 2/3$, we get,

$$\theta(\ell) = 2k + 3 + \frac{2}{3} + 2\delta_1, \quad \nu(\ell) = k - 1 - \frac{2}{3} - \delta_1 - \lambda(3k+1).$$

Thus we have the following table.

ℓ	1	2
k	10	26
$\theta(\ell)$	$23 + \frac{2}{3} + 2\delta_1$	$55 + \frac{2}{3} + 2\delta_1$
$\nu(\ell)$	$9 - \frac{2}{3} - \delta_1 - 31\lambda$	$25 - \frac{2}{3} - \delta_1 - 79\lambda$

If $\lambda < \lambda_0 := \frac{1}{5}$ and $\delta_1 > 0$ is sufficiently small, then $0 < \nu(1) < \nu(2) < 4\nu(1)$. Since

$$\frac{55 + \frac{2}{3} + 2\delta_1}{9 - \frac{2}{3} - \delta_1 - 31\lambda} > \frac{4(23 + \frac{2}{3} + 2\delta_1)}{25 - \frac{2}{3} - \delta_1 - 79\lambda}$$

for all $0 \leq \lambda < \lambda_0$, and $\delta_1 > 0$ can be arbitrarily small, Theorem 5 follows from Theorem 6. \square

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Appendix

In the appendix we shall give the values of $\Delta(\underline{k}, 0, z)$, where $\underline{k} = (k, k+1, k+2)$. Tables in this appendix are organized in the following form.

k	$o(k)$	$A_k(z)$
		$B_k(z)$
		The first two non-zero terms of $R_k(z)$
\underline{k}		$\Delta(\underline{k}, 0, z)$

Approximation polynomials and related determinants in Theorem 1

29	89	$-5500 z^{29} + 264 z^{28} - 432 z^{27} - 432 z^{26} - 5908 z^{25} - 144 z^{24} - 816 z^{23} - 816 z^{22} - 6268 z^{21} - 504 z^{20} - 1152 z^{19} - 1152 z^{18} - 6580 z^{17} - 816 z^{16} - 1440 z^{15} - 1440 z^{14} + 4132 z^{13} - 1632 z^{12} - 864 z^{11} - 864 z^{10} + 4684 z^9 - 1080 z^8 - 336 z^7 - 336 z^6 + 25048 z^5 + 1992 z^4 + 1224 z^3 + 1224 z^2 - 18708 z - 1416$
		$8561 z^{30} + 9917 z^{29} + 18922 z^{28} + 20722 z^{27} + 24653 z^{26} + 27143 z^{25} + 31050 z^{24} + 33516 z^{23} + 36870 z^{22} + 40224 z^{21} + 43404 z^{20} + 46584 z^{19} + 42727 z^{18} + 46075 z^{17} + 41528 z^{16} + 44186 z^{15} + 43421 z^{14} + 46979 z^{13} + 45716 z^{12} + 48776 z^{11} + 40170 z^{10} + 43092 z^9 + 33508 z^8 + 35452 z^7 + 27159 z^6 + 28953 z^5 + 19844 z^4 + 20822 z^3 + 10939 z^2 + 11143 z + 528$
		$-57640 z^{89} + 61912 z^{90}$
		$744 z^{30} - 764 z^{28} + 844 z^{26} - 664 z^{24} + 944 z^{22} - 564 z^{20} + 1044 z^{18} - 464 z^{16} - 444 z^{14} + 1064 z^{12} - 544 z^{10} + 964 z^8 + 444 z^6 - 1064 z^4 - 1388 z^2 + 1628$
30	92	

		$755 z^{31} + 755 z^{30} + 380 z^{29} + 380 z^{28} + 674 z^{27} + 674 z^{26} + 129 z^{25} + 129 z^{24} + 625 z^{23} + 625 z^{22} - 100 z^{21} - 100 z^{20} + 155 z^{19} + 155 z^{18} - 495 z^{17} - 495 z^{16} + 318 z^{15} + 318 z^{14} - 422 z^{13} - 422 z^{12} - 107 z^{11} - 107 z^{10} - 622 z^9 - 622 z^8 - 268 z^7 - 268 z^6 - 553 z^5 - 553 z^4 - 663 z^3 - 663 z^2 - 598 z - 598$ $-5784 z^{92} + 5784 z^{93}$
31	95	$526640 z^{31} - 403916 z^{30} - 1128412 z^{29} - 282452 z^{28} + 377436 z^{27} - 130140 z^{26} - 1216884 z^{25} + 52056 z^{24} + 349696 z^{23} + 265100 z^{22} - 1183892 z^{21} + 508028 z^{20} + 443420 z^{19} + 781804 z^{18} - 1029436 z^{17} + 1085464 z^{16} - 1338644 z^{15} + 860852 z^{14} + 498604 z^{13} + 921584 z^{12} - 825048 z^{11} + 951468 z^{10} + 951468 z^9 + 951468 z^8 + 2249624 z^7 + 50128 z^6 + 412376 z^5 - 10604 z^4 - 2417124 z^3 - 133032 z^2 + 953712 z - 315228$ $1708923 z^{32} + 1514436 z^{31} + 2757186 z^{30} + 2413761 z^{29} + 4598714 z^{28} + 3949701 z^{27} + 5376256 z^{26} + 4497570 z^{25} + 7296661 z^{24} + 6077442 z^{23} + 7720942 z^{22} + 6297837 z^{21} + 8631316 z^{20} + 6946485 z^{19} + 7931494 z^{18} + 6167133 z^{17} + 8925958 z^{16} + 6968556 z^{15} + 8175910 z^{14} + 6253212 z^{13} + 8038642 z^{12} + 6059550 z^{11} + 6430156 z^{10} + 4622415 z^9 + 5576084 z^8 + 3949575 z^7 + 3907170 z^6 + 2659272 z^5 + 2203930 z^4 + 1283310 z^3 + 521312 z^2 + 34251 z + 281488$ $10659096 z^{95} - 8858344 z^{96}$
(29, 30, 31)		$-8520627916960 z^{90} - 15548742756800 z^{89}$
32	98	$736 z^{32} + 64 z^{30} - 68 z^{28} + 96 z^{26} - 36 z^{24} + 128 z^{22} - 4 z^{20} + 160 z^{18} - 148 z^{16} + 32 z^{14} + 164 z^{12} + 296 z^8 - 32 z^6 + 232 z^4 - 96 z^2 + 344$ $-58 z^{33} - 58 z^{32} - 92 z^{31} - 92 z^{30} - 127 z^{29} - 127 z^{28} - 122 z^{27} - 122 z^{26} - 168 z^{25} - 168 z^{24} - 142 z^{23} - 142 z^{22} - 102 z^{21} - 102 z^{20} + 34 z^{19} + 34 z^{18} - z^{17} - z^{16} + 84 z^{15} + 84 z^{14} + 121 z^{13} + 121 z^{12} + 278 z^{11} + 278 z^{10} + 203 z^9 + 203 z^8 + 232 z^7 + 232 z^6 + 145 z^5 + 145 z^4 + 114 z^3 + 114 z^2 + 37 z + 37$ $-64 z^{98} + 64 z^{99}$
33	101	$-64192 z^{33} - 64192 z^{32} - 4800 z^{31} - 4800 z^{30} + 16828 z^{29} - 7936 z^{28} - 8704 z^{27} - 8704 z^{26} + 12732 z^{25} - 12032 z^{24} - 12992 z^{23} - 12992 z^{22} + 8252 z^{21} - 16512 z^{20} - 17664 z^{19} - 17664 z^{18} - 20660 z^{17} + 53632 z^{16} - 5760 z^{15} - 5760 z^{14} - 27964 z^{13} - 3200 z^{12} - 3008 z^{11} - 3008 z^{10} - 47224 z^9 + 2304 z^8 + 2880 z^7 + 2880 z^6 - 40952 z^5 + 8576 z^4 + 9536 z^3 + 9536 z^2 - 9864 z - 59392$ $10958 z^{34} + 9534 z^{33} + 18828 z^{32} + 15740 z^{31} + 28681 z^{30} + 23049 z^{29} + 33158 z^{28} + 24694 z^{27} + 48192 z^{26} + 34544 z^{25} + 52282 z^{24} + 32874 z^{23} + 55378 z^{22} + 28354 z^{21} + 42570 z^{20} + 7258 z^{19} + 45071 z^{18} + 14783 z^{17} + 41812 z^{16} + 740 z^{15} + 32217 z^{14} - 4407 z^{13} + 15854 z^{12} - 31986 z^{11} + 8339 z^{10} - 19437 z^9 + 10440 z^8 - 27784 z^7 + 1009 z^6 - 13535 z^5 + 8906 z^4 - 11990 z^3 + 8709 z^2 - 1547 z + 14704$ $49528 z^{101} - 44792 z^{102}$
(31, 32, 33)		$-42632104148736 z^{96} + 77338989665280 z^{95}$
34	104	$548 z^{34} - 448 z^{32} + 16 z^{30} + 16 z^{28} + 20 z^{26} + 20 z^{24} + 24 z^{22} + 24 z^{20} - 548 z^{18} + 448 z^{16} - 4 z^{14} - 4 z^{12} - 12 z^{10} - 12 z^8 - 20 z^6 - 20 z^4 + 548 z^2 - 448$ $5 z^{35} + 5 z^{34} + 15 z^{33} + 15 z^{32} + 31 z^{31} + 31 z^{30} + 53 z^{29} + 53 z^{28} + 87 z^{27} + 87 z^{26} + 133 z^{25} + 133 z^{24} + 193 z^{23} + 193 z^{22} + 267 z^{21} + 267 z^{20} + 224 z^{19} + 224 z^{18} + 313 z^{17} + 313 z^{16} + 282 z^{15} + 282 z^{14} + 380 z^{13} + 380 z^{12} + 221 z^{11} + 221 z^{10} + 303 z^9 + 303 z^8 + 118 z^7 + 118 z^6 + 164 z^5 + 164 z^4 + 78 z^3 + 78 z^2 + 109 z + 109$ $-8 z^{104} + 8 z^{105}$
35	112	$4 z^{35} - 4 z^{34} - 4 z^{33} + 4 z^{32} - 4 z^{19} + 4 z^{18} + 4 z^{17} - 4 z^{16} + 4 z^3 - 4 z^2 - 4 z + 4$ $-z^{20} - z^{16} - 2 z^{12} - 2 z^8 - z^4 - 1$ $8 z^{112} - 16 z^{113}$
36	112	$4 z^{35} + 8 z^{34} - 4 z^{33} - 4 z^{32} - 4 z^{20} - 4 z^{19} - 8 z^{18} + 4 z^{17} + 8 z^{16} + 4 z^4 + 4 z^3 + 8 z^2 - 4 z - 8$

		$z^{37} + z^{36} + 2z^{35} + 2z^{34} + 3z^{33} + 3z^{32} + 4z^{31} + 4z^{30} + 6z^{29} + 6z^{28} + 8z^{27} + 8z^{26} + 10z^{25} + 10z^{24} + 12z^{23} + 12z^{22} + 15z^{21} + 14z^{20} + 15z^{19} + 15z^{18} + 17z^{17} + 16z^{16} + 16z^{15} + 16z^{14} + 18z^{13} + 16z^{12} + 14z^{11} + 14z^{10} + 14z^9 + 12z^8 + 8z^7 + 8z^6 + 8z^5 + 7z^4 + 5z^3 + 5z^2 + 4z + 3$
		$-16z^{112} + 8z^{113}$
(34, 35, 36)		$32z^{105} - 32z^{104}$
43	131	$4z^{43} - 8z^{42} + 8z^{40} - 8z^{39} + 8z^{38} - 8z^{36} + 4z^{35}$ $-z^{44} + z^{43} - z^{36} + z^{35} - 2z^{28} + 2z^{27} - 2z^{20} + 2z^{19} - z^{12} + z^{11} - z^4 + z^3$ $-8z^{131} + 24z^{132}$
44	134	$64z^{44} - 88z^{42} - 44z^{40} + 88z^{38} - 4z^{36} - 4z^{32} + 4z^{20} + 4z^{16} + 4z^{12} + 4z^8 + 4z^4 + 4$ $-15z^{45} - 15z^{44} - 8z^{43} - 8z^{42} - 4z^{41} - 4z^{40} - 13z^{37} - 13z^{36} - 4z^{35} - 4z^{34} + z^{33} + z^{32} + 6z^{31} + 6z^{30} - 22z^{29} - 22z^{28} - 6z^{27} - 6z^{26} + 2z^{25} + 2z^{24} + 10z^{23} + 10z^{22} - 18z^{21} - 18z^{20} - 2z^{19} - 2z^{18} + 4z^{17} + 4z^{16} + 10z^{15} + 10z^{14} - 5z^{13} - 5z^{12} + 2z^{11} + 2z^{10} + 4z^9 + 4z^8 + 6z^7 + 6z^6 - 10z^5 - 10z^4 - 4z^3 - 4z^2 - 2z - 2$ $-176z^{134} + 176z^{135}$
45	137	$304192z^{45} + 262504z^{44} - 72072z^{43} - 72072z^{42} - 40788z^{41} + 35640z^{40} + 146432z^{39} + 146432z^{38} + 84828z^{37} + 77880z^{36} + 128128z^{35} + 128128z^{34} + 69956z^{33} + 63008z^{32} + 116688z^{31} + 116688z^{30} + 106392z^{29} + 106392z^{28} + 96096z^{27} + 96096z^{26} + 85800z^{25} + 85800z^{24} + 75504z^{23} + 75504z^{22} + 109652z^{21} + 116600z^{20} + 38896z^{19} + 38896z^{18} + 69612z^{17} + 76560z^{16} - 4576z^{15} - 4576z^{14} + 53596z^{13} + 60544z^{12} + 6864z^{11} + 6864z^{10} + 61604z^9 + 68552z^8 + 11440z^7 + 11440z^6 + 62748z^5 + 69696z^4 + 9152z^3 + 9152z^2 + 57028z + 63976$ $-57215z^{46} - 102271z^{45} - 130028z^{44} - 145626z^{43} - 190986z^{42} - 229398z^{41} - 260744z^{40} - 285142z^{39} - 385343z^{38} - 469911z^{37} - 525496z^{36} - 565448z^{35} - 628583z^{34} - 683033z^{33} - 700980z^{32} - 710242z^{31} - 835054z^{30} - 932074z^{29} - 949350z^{28} - 938834z^{27} - 966634z^{26} - 980538z^{25} - 927114z^{24} - 859794z^{23} - 928550z^{22} - 969514z^{21} - 918722z^{20} - 840138z^{19} - 825524z^{18} - 800488z^{17} - 715518z^{16} - 620126z^{15} - 616995z^{14} - 601705z^{13} - 531364z^{12} - 448864z^{11} - 396040z^{10} - 339742z^9 - 258032z^8 - 172848z^7 - 169960z^6 - 156650z^5 - 117726z^4 - 68380z^3 - 58456z^2 - 45058z - 24266$ $180648z^{137} + 288392z^{138}$
(43, 44, 45)		$247104z^{132} - 247104z^{131}$
49	149	$-30924z^{49} + 7224z^{48} + 8904z^{47} + 8904z^{46} + 74616z^{45} - 1680z^{44} - 1176z^{43} - 1176z^{42} - 64704z^{41} + 11592z^{40} + 13272z^{39} + 13272z^{38} + 45876z^{37} + 7728z^{36} + 7728z^{35} + 7728z^{34} - 89412z^{33} + 25032z^{32} + 25704z^{31} + 25704z^{30} + 24192z^{29} + 24192z^{28} + 22680z^{27} + 22680z^{26} - 109080z^{25} + 43512z^{24} + 42168z^{23} + 42168z^{22} + 5532z^{21} + 43680z^{20} + 39648z^{19} + 39648z^{18} + 172020z^{17} + 32144z^{16} + 10416z^{15} + 10416z^{14} - 22692z^{13} + 15456z^{12} + 14952z^{11} + 14952z^{10} - 18660z^9 + 19488z^8 + 18480z^7 + 18480z^6 - 15636z^5 + 22512z^4 + 21000z^3 + 21000z^2 + 116628z + 2184$ $17547z^{50} + 19199z^{49} + 33722z^{48} + 32350z^{47} + 37321z^{46} + 39113z^{45} + 42950z^{44} + 43608z^{43} + 98001z^{42} + 98351z^{41} + 143378z^{40} + 134362z^{39} + 190369z^{38} + 179617z^{37} + 226090z^{36} + 205804z^{35} + 279265z^{34} + 254177z^{33} + 314912z^{32} + 277098z^{31} + 305202z^{30} + 269726z^{29} + 292874z^{28} + 252442z^{27} + 306568z^{26} + 265324z^{25} + 310126z^{24} + 259558z^{23} + 266266z^{22} + 222110z^{21} + 227222z^{20} + 181470z^{19} + 195921z^{18} + 156329z^{17} + 167882z^{16} + 125392z^{15} + 88177z^{14} + 60499z^{13} + 29290z^{12} + 7618z^{11} + 16837z^{10} + 3803z^9 + 11636z^8 - 2784z^7 - 1788z^6 - 7150z^5 - 6238z^4 - 11684z^3 - 1502z^2 - 4036z + 4340$ $228888z^{149} - 195624z^{150}$
		$-120z^{50} - 120z^{48} - 36z^{46} - 36z^{44} - 120z^{42} - 120z^{40} - 48z^{34} - 48z^{32} + 108z^{30} + 108z^{28} + 96z^{26} + 96z^{24} + 288z^{22} + 288z^{20} + 932z^{18} - 296z^{16} + 36z^{14} + 36z^{12} + 72z^{10} + 72z^8 + 108z^6 + 108z^4 + 156z^2 + 156$
50	152	

		$179z^{51} + 179z^{50} + 230z^{49} + 230z^{48} + 433z^{47} + 433z^{46} + 481z^{45} + 481z^{44} + 866z^{43} + 866z^{42} + 974z^{41} + 974z^{40} + 1329z^{39} + 1329z^{38} + 1317z^{37} + 1317z^{36} + 1561z^{35} + 1561z^{34} + 1447z^{33} + 1447z^{32} + 1544z^{31} + 1544z^{30} + 1238z^{29} + 1238z^{28} + 1230z^{27} + 1230z^{26} + 906z^{25} + 906z^{24} + 844z^{23} + 844z^{22} + 430z^{21} + 430z^{20} + 353z^{19} + 353z^{18} - z^{17} - z^{16} - 36z^{15} - 36z^{14} - 366z^{13} - 366z^{12} - 290z^{11} - 290z^{10} - 422z^9 - 422z^8 - 211z^7 - 211z^6 - 271z^5 - 271z^4 - 149z^3 - 149z^2 - 152z - 152$ $-96z^{152} + 96z^{153}$
51	160	$4z^{19} - 4z^{18} - 4z^{17} + 4z^{16}$ $z^{52} + z^{48} + 2z^{44} + 2z^{40} + 2z^{36} + 2z^{32} + 2z^{28} + 2z^{24} + 2z^{20} + 2z^{16} + 2z^{12} + 2z^8 + z^4 + 1$ $8z^{160} - 16z^{161}$
(49, 50, 51)		$-30517344z^{150} + 35706528z^{149}$

Approximation polynomials and related determinants in Theorem 2

29	89	$8838z^{29} - 103526z^{28} + 48074z^{27} + 249006z^{26} - 36267z^{25} - 222096z^{24} - 98200z^{23} - 42694z^{22} + 236901z^{21} + 813160z^{20} + 82188z^{19} + 105172z^{18} - 76623z^{17} - 341910z^{16} - 29528z^{15} + 101076z^{14} + 42482z^{13} + 214028z^{12} - 7828z^{11} - 293424z^{10} - 61460z^9 + 84522z^8 + 90968z^7 - 67100z^6 - 221774z^5 - 749016z^4 - 82002z^3 + 59754z^2 + 157247z + 372960$ $249196z^{30} + 197392z^{29} + 504078z^{28} - 11390z^{27} + 8214z^{26} - 17793z^{25} - 222796z^{24} - 40936z^{23} - 71854z^{22} - 31047z^{21} + 355216z^{20} + 158208z^{19} - 251360z^{18} - 245645z^{17} - 403198z^{16} - 44440z^{15} - 75264z^{14} + 35828z^{13} - 391672z^{12} - 126276z^{11} + 392512z^{10} + 300274z^9 + 635406z^8 + 79336z^7 + 155388z^6 - 30926z^5 + 185780z^4 + 8222z^3 - 417854z^2 - 180793z - 348900$ $-341396z^{89} - 904548z^{90}$
30	92	$284151z^{30} - 303211z^{29} + 34473z^{28} + 230361z^{27} - 272325z^{26} + 122016z^{25} + 180112z^{24} - 334273z^{23} + 109371z^{22} + 36208z^{21} - 196837z^{20} + 184636z^{19} + 92877z^{18} - 222239z^{17} + 170339z^{16} + 15686z^{15} - 212872z^{14} + 178126z^{13} + 20616z^{12} - 143468z^{11} + 163829z^{10} - 56575z^9 - 134101z^8 + 230702z^7 - 66928z^6 + 9672z^5 + 97210z^4 - 157573z^3 - 40095z^2 + 125240z - 109296$ $-59086z^{31} + 15283z^{30} - 74369z^{29} + 60109z^{28} + 28737z^{27} - 15235z^{26} + 39866z^{25} - 14610z^{24} - 15469z^{23} + 16375z^{22} - 86676z^{21} + 36259z^{20} + 116870z^{19} - 82905z^{18} + 67053z^{17} + 31655z^{16} - 103912z^{15} + 26480z^{14} + 172732z^{13} - 187648z^{12} - 74152z^{11} + 153851z^{10} - 249333z^9 + 87897z^8 + 154690z^7 - 163750z^6 - 19530z^5 + 191628z^4 - 94209z^3 - 18717z^2 + 204414z - 206502$ $379224z^{92} + 172268z^{93}$
31	95	$-4225266z^{31} + 1316319z^{30} + 2983440z^{29} - 3968815z^{28} + 1611693z^{27} + 1846233z^{26} - 4256835z^{25} + 2557108z^{24} + 1911015z^{23} - 3531513z^{22} + 3405999z^{21} + 155295z^{20} - 3334914z^{19} + 2859543z^{18} - 181131z^{17} - 2388117z^{16} + 2999370z^{15} - 581472z^{14} - 2191518z^{13} + 2452914z^{12} - 917898z^{11} - 1244721z^{10} + 2592741z^9 - 1622013z^8 - 1446126z^7 + 2251806z^6 - 2410077z^5 + 236958z^4 + 2263671z^3 - 2142621z^2 + 587307z + 1506461$ $303774z^{32} + 398004z^{31} - 205521z^{30} + 755412z^{29} - 338283z^{28} - 77559z^{27} + 87615z^{26} - 492033z^{25} + 285082z^{24} - 79797z^{23} + 281151z^{22} + 665469z^{21} - 1280281z^{20} - 3960z^{19} + 863013z^{18} - 1755093z^{17} + 770027z^{16} + 705864z^{15} - 1921572z^{14} + 387582z^{13} + 1832330z^{12} - 1503042z^{11} + 429153z^{10} + 2481123z^9 - 2516947z^8 + 797970z^7 + 1745100z^6 - 2355387z^5 + 325532z^4 + 1275255z^3 - 1951479z^2 + 45261z + 1580655$ $5421516z^{95} - 3588972z^{96}$
(29, 30, 31)		$-127293061031881800z^{90} - 47224605589276632z^{89}$

32	98	$-22588299 z^{32} + 13357080 z^{31} - 660150 z^{30} - 11245880 z^{29} + 13273201 z^{28} - 4213080 z^{27} - 10169442 z^{26} + 14802080 z^{25} - 4524244 z^{24} - 4588680 z^{23} + 13870419 z^{22} - 8577000 z^{21} - 3890070 z^{20} + 10892280 z^{19} - 8706669 z^{18} - 1315560 z^{17} + 10323339 z^{16} - 8303040 z^{15} - 616950 z^{14} + 7345200 z^{13} - 8432709 z^{12} + 1957560 z^{11} + 6776259 z^{10} - 9789480 z^9 + 2206179 z^8 + 2264400 z^7 - 9081531 z^6 + 6965040 z^5 + 2002044 z^4 - 6780360 z^3 + 6752958 z^2 + 61720 z + 4831822$
		$1760400 z^{33} + 449991 z^{32} + 1533720 z^{31} + 922782 z^{30} - 1734360 z^{29} + 1227135 z^{28} - 974760 z^{27} - 823668 z^{26} + 1069520 z^{25} - 1197466 z^{24} + 2592120 z^{23} + 473565 z^{22} - 4485800 z^{21} + 2729386 z^{20} - 1450200 z^{19} - 4290933 z^{18} + 5043160 z^{17} - 1619519 z^{16} - 6351840 z^{15} + 5718510 z^{14} + 2455600 z^{13} - 5795075 z^{12} + 8621640 z^{11} + 1149129 z^{10} - 7986200 z^9 + 8595727 z^8 + 484800 z^7 - 7667985 z^6 + 5206240 z^5 - 338900 z^4 - 7833240 z^3 + 6179112 z^2 + 663720 z + 4080948$
		$-37889148 z^{98} + 9889572 z^{99}$
33	101	$-176064633 z^{33} - 85665591 z^{32} + 74750757 z^{31} - 47272970 z^{30} + 21499188 z^{29} + 14694666 z^{28} - 95780913 z^{27} + 43755587 z^{26} + 68776500 z^{25} - 4037217 z^{24} + 64432656 z^{23} + 27953142 z^{22} - 75272931 z^{21} + 19707900 z^{20} + 4403736 z^{19} - 33320676 z^{18} + 62965368 z^{17} + 7176237 z^{16} - 53028738 z^{15} + 18240612 z^{14} - 16373169 z^{13} - 11076483 z^{12} + 61498080 z^{11} - 43239204 z^{10} - 55583784 z^9 - 12887979 z^8 - 46376811 z^7 - 4446624 z^6 + 58106862 z^5 - 9556881 z^4 + 3266673 z^3 + 35390365 z^2 + 47644845 z + 47235177$
		$29638536 z^{34} + 24799239 z^{33} + 29661303 z^{32} + 9226737 z^{31} + 15614334 z^{30} + 1923930 z^{29} - 24820692 z^{28} - 11807025 z^{27} - 5265061 z^{26} - 5139888 z^{25} + 43840173 z^{24} + 25801494 z^{23} - 53349362 z^{22} - 20883027 z^{21} - 5941698 z^{20} - 35963706 z^{19} - 3232646 z^{18} + 17530458 z^{17} - 72115791 z^{16} - 3288378 z^{15} + 68504836 z^{14} - 9686523 z^{13} + 50244111 z^{12} + 55708260 z^{11} - 11129384 z^{10} + 14958006 z^9 + 44808585 z^8 - 49038303 z^7 - 28482668 z^6 + 19260636 z^5 - 58513881 z^4 + 205749 z^3 + 9770433 z^2 + 39546873 z + 55820127$
		$-332338020 z^{101} - 144335068 z^{102}$
(31, 32, 33)		$-445634331745438289520 z^{96} + 417178155931626734568 z^{95}$
34	104	$-399703431 z^{34} + 1087916661 z^{33} - 1630720744 z^{32} - 123158126 z^{31} - 21796076 z^{30} + 189852624 z^{29} - 422979590 z^{28} + 46370805 z^{27} + 286887150 z^{26} - 644304791 z^{25} + 995887336 z^{24} + 26439608 z^{23} - 84276316 z^{22} + 67661308 z^{21} + 23651188 z^{20} - 47593026 z^{19} + 55252839 z^{18} - 61203883 z^{17} - 32396137 z^{16} + 103424620 z^{15} - 63992303 z^{14} + 4287153 z^{13} + 91016151 z^{12} - 54334722 z^{11} - 247162617 z^{10} + 629737225 z^9 - 1008730988 z^8 - 52474214 z^7 + 164818441 z^6 - 223558699 z^5 + 266169085 z^4 + 73506901 z^3 + 179099632 z^2 - 630460649 z + 1041984688$
		$-94512652 z^{35} + 266646659 z^{34} - 490549293 z^{33} + 857095194 z^{32} + 108641546 z^{31} - 38038978 z^{30} - 12932166 z^{29} - 194804864 z^{28} + 54188017 z^{27} - 85203872 z^{26} - 50724817 z^{25} + 364466056 z^{24} - 113721868 z^{23} - 151523710 z^{22} + 505358802 z^{21} - 756027622 z^{20} + 2616640 z^{19} + 24294241 z^{18} + 176360093 z^{17} - 629232539 z^{16} + 29058172 z^{15} + 224824483 z^{14} - 663863213 z^{13} + 1293732477 z^{12} - 45638242 z^{11} + 33431723 z^{10} - 139780043 z^9 + 284577000 z^8 + 108864730 z^7 - 175863023 z^6 + 375097749 z^5 - 839815275 z^4 - 44561283 z^3 + 172085310 z^2 - 459202523 z + 910810930$
		$-1702388928 z^{104} - 837017980 z^{105}$
35	107	$-1046716196 z^{35} - 1705139966 z^{34} + 1328534848 z^{33} + 209186103 z^{32} - 579850958 z^{31} + 218235941 z^{30} - 105588449 z^{29} + 39130547 z^{28} + 948112085 z^{27} + 821966732 z^{26} - 715563620 z^{25} - 86195686 z^{24} - 108468120 z^{23} - 221335144 z^{22} + 349140044 z^{21} - 205764255 z^{20} - 57327118 z^{19} + 296572370 z^{18} - 344630703 z^{17} + 92536691 z^{16} + 271430792 z^{15} - 441926838 z^{14} + 256544717 z^{13} + 73948784 z^{12} - 877888846 z^{11} - 762881152 z^{10} + 620094166 z^9 + 185407898 z^8 + 208876518 z^7 + 575933089 z^6 - 448129406 z^5 - 97527619 z^4 + 781030829 z^3 + 678076638 z^2 - 536082374 z - 98863393$

		$144914334 z^{36} + 297095560 z^{35} + 1169289678 z^{34} - 426372626 z^{33} - 863497319 z^{32} +$ $361540034 z^{31} - 434779223 z^{30} - 368826463 z^{29} + 890265147 z^{28} + 6887337 z^{27} -$ $155113820 z^{26} + 542917024 z^{25} - 518948038 z^{24} - 879518516 z^{23} - 316730730 z^{22} +$ $557695048 z^{21} - 296928951 z^{20} + 74727036 z^{19} - 279021182 z^{18} - 506772977 z^{17} +$ $1040291399 z^{16} + 712099152 z^{15} + 918083242 z^{14} - 214816021 z^{13} - 804575800 z^{12} -$ $106173246 z^{11} + 406912124 z^{10} - 44306314 z^9 - 497884300 z^8 + 1563142 z^7 -$ $1043442955 z^6 - 18436418 z^5 + 1246202671 z^4 + 261494749 z^3 + 905060170 z^2 -$ $152646960 z - 822091223$ $-550045468 z^{107} - 1597702416 z^{108}$
36	110	$-72679207 z^{36} - 60042465 z^{35} - 27328323 z^{34} + 28175850 z^{33} + 38242280 z^{32} -$ $1018170 z^{31} - 19881498 z^{30} - 75330 z^{29} + 64690622 z^{28} + 35278740 z^{27} + 28066812 z^{26} -$ $14776020 z^{25} - 50238100 z^{24} - 8127540 z^{23} + 8674680 z^{22} - 1006830 z^{21} + 551847 z^{20} +$ $7718085 z^{19} - 12637842 z^{18} - 1883655 z^{17} + 10669710 z^{16} - 9245745 z^{15} + 6795732 z^{14} +$ $6166935 z^{13} - 45339792 z^{12} - 32359095 z^{11} - 28120197 z^{10} + 15981300 z^9 +$ $56558841 z^8 + 19362645 z^7 + 6299418 z^6 - 10006335 z^5 + 27751201 z^4 + 27999270 z^3 +$ $23655654 z^2 - 12371130 z - 34995848$ $3546180 z^{37} + 24463035 z^{36} + 36719325 z^{35} + 40784937 z^{34} - 33465960 z^{33} -$ $43519224 z^{32} - 7654500 z^{31} - 35415090 z^{30} + 22383540 z^{29} + 35479690 z^{28} -$ $5341950 z^{27} + 29684478 z^{26} - 8292780 z^{25} - 67702526 z^{24} - 19643310 z^{23} -$ $9206988 z^{22} + 257580 z^{21} + 18201351 z^{20} - 5600745 z^{19} - 45297798 z^{18} + 24472935 z^{17} +$ $84518202 z^{16} + 34033365 z^{15} + 57933060 z^{14} - 29645595 z^{13} - 68674026 z^{12} +$ $9503325 z^{11} + 11405619 z^{10} - 15313860 z^9 - 26194131 z^8 - 28855035 z^7 - 53742390 z^6 +$ $38598525 z^5 + 88270743 z^4 + 28094040 z^3 + 43862046 z^2 - 28627020 z - 56682498$ $-31540680 z^{110} - 98505000 z^{111}$
(34, 35, 36)		$-10749530318906122061682776 z^{105} + 39437465990135596375969920 z^{104}$
38	116	$-328392625 z^{38} + 290993025 z^{37} - 106716958 z^{36} - 80099400 z^{35} + 146334448 z^{34} -$ $82673700 z^{33} - 10372071 z^{32} + 109928850 z^{31} + 160680958 z^{30} - 236200200 z^{29} +$ $178868184 z^{28} - 25575900 z^{27} - 133420172 z^{26} + 154610400 z^{25} - 68735318 z^{24} -$ $68040750 z^{23} + 90264865 z^{22} - 26269275 z^{21} - 64890392 z^{20} + 97028775 z^{19} -$ $58008848 z^{18} - 40843875 z^{17} + 125617990 z^{16} - 79415775 z^{15} - 137346362 z^{14} +$ $191098125 z^{13} - 157753650 z^{12} + 30459150 z^{11} + 147917207 z^{10} - 161086425 z^9 +$ $76506261 z^8 + 42485775 z^7 + 119836337 z^6 - 146144100 z^5 + 121742572 z^4 -$ $26396550 z^3 - 74539424 z^2 + 109114050 z - 112978051$ $-9403500 z^{39} + 99652845 z^{38} - 58716225 z^{37} + 37194802 z^{36} - 44106150 z^{35} +$ $13053908 z^{34} + 9717150 z^{33} - 87002903 z^{32} + 120547500 z^{31} - 88922450 z^{30} -$ $15160650 z^{29} + 146537548 z^{28} - 160122600 z^{27} - 75053458 z^{26} + 189482250 z^{25} -$ $103836420 z^{24} - 938400 z^{23} + 87715453 z^{22} - 87401325 z^{21} - 85839656 z^{20} +$ $176370225 z^{19} + 52126768 z^{18} - 167935725 z^{17} + 205536492 z^{16} - 158374125 z^{15} +$ $13195820 z^{14} + 116660925 z^{13} - 71477426 z^{12} + 1417950 z^{11} - 43444389 z^{10} +$ $16305675 z^9 - 84638221 z^8 + 124726875 z^7 + 67343275 z^6 - 135781050 z^5 +$ $126170316 z^4 - 97316700 z^3 - 8126386 z^2 + 76755450 z - 33574889$ $-130668712 z^{116} + 529192388 z^{117}$
39	119	$-149119706349 z^{39} + 63707949546 z^{38} - 26395317696 z^{37} - 36247180542 z^{36} +$ $59443318176 z^{35} - 31786065503 z^{34} - 9258370371 z^{33} + 37607627380 z^{32} +$ $80661854706 z^{31} - 57012122504 z^{30} + 62282372040 z^{29} - 7098995408 z^{28} -$ $61235049540 z^{27} + 50983686882 z^{26} - 20606379006 z^{25} - 28591419162 z^{24} +$ $35781687273 z^{23} - 9322212819 z^{22} - 30043911930 z^{21} + 39553662819 z^{20} -$ $20042522118 z^{19} - 18860076597 z^{18} + 53019807420 z^{17} - 38017902921 z^{16} -$ $67954416972 z^{15} + 44264940214 z^{14} - 55991529852 z^{13} + 15502477726 z^{12} +$ $68947673487 z^{11} - 41919255682 z^{10} + 22755960363 z^9 + 17094442241 z^8 +$ $57342356871 z^7 - 33076245296 z^6 + 43263248676 z^5 - 12548417012 z^4 -$ $34528437732 z^3 + 38395278251 z^2 - 43071249867 z + 10796816534$

		$ \begin{aligned} & 3019569924 z^{40} + 46348505073 z^{39} + 12526842752 z^{38} + 10272887556 z^{37} - \\ & 25924554292 z^{36} + 1310254080 z^{35} - 18755197493 z^{34} - 35923872231 z^{33} + \\ & 58445245834 z^{32} - 31003013634 z^{31} + 5501615364 z^{30} + 62566901664 z^{29} - \\ & 71660413140 z^{28} - 45616511226 z^{27} + 49880906186 z^{26} - 32766551160 z^{25} - \\ & 23248226440 z^{24} + 38184226665 z^{23} - 46360350849 z^{22} - 41824933494 z^{21} + \\ & 97154733705 z^{20} + 38120227062 z^{19} - 17090569745 z^{18} + 76837413774 z^{17} - \\ & 60618822959 z^{16} - 4884457938 z^{15} + 41319863530 z^{14} - 24244285020 z^{13} - \\ & 24745519454 z^{12} - 21728806557 z^{11} - 28922373896 z^{10} - 33575376855 z^9 + \\ & 69559600537 z^8 + 41361916065 z^7 - 9475846394 z^6 + 44214361152 z^5 - \\ & 34357388522 z^4 - 10977837426 z^3 + 18941010169 z^2 - 9408704265 z - 13383501602 \\ & - 484474868376 z^{119} - 461974967404 z^{120} \end{aligned} $
40	122	$ \begin{aligned} & 91864726291 z^{40} + 319280781460 z^{39} - 224030950487 z^{38} + 27374191880 z^{37} + \\ & 292533708140 z^{36} - 292674605820 z^{35} - 1510481652 z^{34} + 44669145100 z^{33} - \\ & 334085225207 z^{32} - 134799607080 z^{31} + 317018563878 z^{30} - 254112973440 z^{29} - \\ & 145256148136 z^{28} + 252753057880 z^{27} - 182489746860 z^{26} + 36049279040 z^{25} + \\ & 258332973123 z^{24} - 158528733880 z^{23} - 48928115708 z^{22} + 193350332700 z^{21} - \\ & 140746785575 z^{20} + 38800707260 z^{19} + 159559557623 z^{18} - 333737842620 z^{17} + \\ & 34029855687 z^{16} + 103226012960 z^{15} - 240864919868 z^{14} + 290346874040 z^{13} + \\ & 113028048961 z^{12} - 180497459260 z^{11} + 172178300702 z^{10} - 72252586560 z^9 - \\ & 256768685827 z^8 - 94832110460 z^7 + 161086577681 z^6 - 220632804400 z^5 - \\ & 82608922550 z^4 + 163568737100 z^3 - 139940524770 z^2 + 286924169860 z + \\ & 198899712593 \\ & 58079840240 z^{41} + 20779678345 z^{40} + 103515938500 z^{39} - 17011134207 z^{38} - \\ & 147808517040 z^{37} + 59923127422 z^{36} - 212289811620 z^{35} - 13631116930 z^{34} + \\ & 321426076900 z^{33} - 153779160721 z^{32} + 233225695720 z^{31} + 128766125602 z^{30} - \\ & 434954929680 z^{29} + 207808787358 z^{28} + 64712750720 z^{27} - 349503491512 z^{26} - \\ & 73802320960 z^{25} + 81918338683 z^{24} - 322745890720 z^{23} + 126471217758 z^{22} + \\ & 533202716180 z^{21} - 303556927977 z^{20} + 206915469740 z^{19} + 290555338185 z^{18} - \\ & 398457916780 z^{17} + 132114286577 z^{16} + 39320376200 z^{15} - 306305865594 z^{14} - \\ & 132039615160 z^{13} + 68052119533 z^{12} - 229173884420 z^{11} + 57726516830 z^{10} + \\ & 402305026120 z^9 - 200625842251 z^8 + 162582080740 z^7 + 207397909869 z^6 - \\ & 204794211720 z^5 + 97428157174 z^4 - 17980049220 z^3 - 177803934800 z^2 - \\ & 84951726820 z + 25430355927 \\ & - 537183155528 z^{122} - 703138211928 z^{123} \end{aligned} $
(38, 39, 40)		$1070663185232794219680724801040 z^{117} - 383714096776182783428173226848 z^{116}$
43	131	$ \begin{aligned} & 654719760512 z^{43} - 2821346833612 z^{42} - 94694672644 z^{41} + 2479488890145 z^{40} - \\ & 512912959904 z^{39} + 2552059180057 z^{38} + 90318251584 z^{37} - 540036131793 z^{36} - \\ & 332932250408 z^{35} + 689404484714 z^{34} - 233019909196 z^{33} - 1816808882738 z^{32} + \\ & 350569737040 z^{31} - 298843735826 z^{30} + 80388796672 z^{29} - 759331507046 z^{28} - \\ & 193863522676 z^{27} + 98917002890 z^{26} + 239024931320 z^{25} + 507105169032 z^{24} + \\ & 55088360924 z^{23} + 45426928350 z^{22} - 450655798776 z^{21} - 92389707567 z^{20} + \\ & 127506668620 z^{19} - 104589492891 z^{18} + 423189766836 z^{17} - 553853895119 z^{16} - \\ & 211431381744 z^{15} - 129812189536 z^{14} - 130698692896 z^{13} + 643148743452 z^{12} - \\ & 340716989224 z^{11} + 2775000173564 z^{10} - 160997824948 z^9 - 2755147607899 z^8 + \\ & 301462445928 z^7 - 2157483293747 z^6 + 366504279088 z^5 + 596005489469 z^4 + \\ & 138760078784 z^3 - 725373895320 z^2 - 128931956988 z + 2197279449742 \end{aligned} $

		$1600354283224 z^{44} + 229103252712 z^{43} - 1515378673264 z^{42} - 223609710724 z^{41} + 626899969097 z^{40} - 411335257848 z^{39} + 1782268460381 z^{38} + 514927759856 z^{37} - 1635050608811 z^{36} + 471980486624 z^{35} - 2021350873784 z^{34} - 698337174500 z^{33} + 2025446830184 z^{32} - 67928635024 z^{31} + 2924131042610 z^{30} - 13100256640 z^{29} - 1559320311274 z^{28} - 450365666348 z^{27} - 1826792092706 z^{26} + 681361714048 z^{25} + 1581036201434 z^{24} + 469409501316 z^{23} - 2128757496492 z^{22} - 625900977864 z^{21} + 1072970184197 z^{20} - 161130674252 z^{19} + 4404057533735 z^{18} + 2130376028 z^{17} - 3832527512063 z^{16} - 342823093552 z^{15} - 1230015018030 z^{14} + 496838785344 z^{13} + 1676061259314 z^{12} + 420162704880 z^{11} - 2827316268964 z^{10} - 350037673132 z^9 + 1078126371345 z^8 - 203066232000 z^7 + 4111288791141 z^6 - 26920017216 z^5 - 2150284069005 z^4 + 30602791784 z^3 - 1803737859838 z^2 - 50999731780 z + 1712645539000$
		$1922418947808 z^{131} + 6043431728416 z^{132}$
44	134	$-75947033832 z^{44} - 5579107092 z^{43} - 104238012156 z^{42} + 4739827911 z^{41} + 105950111578 z^{40} + 4937188815 z^{39} + 95161460838 z^{38} - 1864833711 z^{37} + 82640354966 z^{36} + 1664230542 z^{35} + 19169079328 z^{34} - 1874055534 z^{33} - 110419687504 z^{32} - 2242997406 z^{31} + 2455649268 z^{30} - 41909634 z^{29} - 71433888516 z^{28} + 614444454 z^{27} + 4379526516 z^{26} - 773138496 z^{25} + 38156981404 z^{24} + 947504922 z^{23} - 11826042596 z^{22} + 1292556447 z^{21} - 355903374 z^{20} - 2056683525 z^{19} + 16466899058 z^{18} - 1263948465 z^{17} - 53825333390 z^{16} + 1616137536 z^{15} - 20478872416 z^{14} + 74222772 z^{13} + 120456198304 z^{12} + 4671106596 z^{11} + 104403966276 z^{10} - 3272846733 z^9 - 123793016934 z^8 - 5059153749 z^7 - 66892041266 z^6 + 393363555 z^5 - 64852742270 z^4 + 298156560 z^3 - 41110831292 z^2 + 2510715714 z + 140992632208$
		$2216770296 z^{45} + 57129877336 z^{44} - 2728300848 z^{43} - 60713752852 z^{42} + 2166681231 z^{41} + 40970060514 z^{40} + 2967191643 z^{39} + 77208074222 z^{38} - 4780052277 z^{37} - 97090594078 z^{36} - 2387742864 z^{35} - 96526339780 z^{34} + 5426834712 z^{33} + 90857543716 z^{32} + 2175781374 z^{31} + 132551744076 z^{30} - 167355630 z^{29} - 46431083436 z^{28} - 2857116366 z^{27} - 61093058028 z^{26} - 1027769106 z^{25} + 6007230904 z^{24} - 935469756 z^{23} - 111414475152 z^{22} + 2497628523 z^{21} + 96742259546 z^{20} + 5218568745 z^{19} + 185829175462 z^{18} - 3930963153 z^{17} - 131339449078 z^{16} - 2434706274 z^{15} - 36798869300 z^{14} + 377423982 z^{13} + 10056749292 z^{12} - 2820430860 z^{11} - 128752306132 z^{10} + 1823523351 z^9 + 71211869794 z^8 + 6026206563 z^7 + 164593057134 z^6 - 2461736979 z^5 - 56421410130 z^4 - 3939916266 z^3 - 63767607912 z^2 + 3454515480 z + 25963043060$
		$-512186985504 z^{134} - 522891150768 z^{135}$
45	137	$541293377520 z^{45} - 968893495884 z^{44} - 1198079333952 z^{43} + 447173795391 z^{42} + 777341459536 z^{41} - 88091856171 z^{40} + 1040857866600 z^{39} + 15100602735 z^{38} - 1111190897800 z^{37} + 1538190196584 z^{36} + 371502990856 z^{35} - 964937789790 z^{34} + 168304420664 z^{33} - 170359771302 z^{32} - 770970946800 z^{31} + 147538176606 z^{30} + 383820068304 z^{29} - 1091557612278 z^{28} + 246926658864 z^{27} + 587156627544 z^{26} - 516091336808 z^{25} + 390068663154 z^{24} + 288431248168 z^{23} - 1057237233597 z^{22} + 510827218944 z^{21} + 346537536069 z^{20} - 713212429168 z^{19} + 937402306665 z^{18} - 174655591352 z^{17} - 1070279230902 z^{16} + 671183463200 z^{15} - 291468004068 z^{14} - 735491160272 z^{13} + 1861353055764 z^{12} + 812032969152 z^{11} - 981007879581 z^{10} - 176504582496 z^9 - 240996892647 z^8 - 1161587791976 z^7 + 1040368757721 z^6 + 435701866840 z^5 - 1674922778490 z^4 + 365855876968 z^3 + 103783298334 z^2 - 50206939352 z + 1047984167376$

		$ \begin{aligned} & 375826069824 z^{46} + 389037204976 z^{45} - 51783796104 z^{44} - 564345721792 z^{43} - \\ & 316470592089 z^{42} + 407337506496 z^{41} + 65331297489 z^{40} + 607967063384 z^{39} + \\ & 718385445921 z^{38} - 941731846600 z^{37} - 402580388634 z^{36} - 318233580952 z^{35} - \\ & 1105125087036 z^{34} + 1260902030248 z^{33} + 928573409322 z^{32} - 214019408016 z^{31} - \\ & 139378083822 z^{30} + 240560920848 z^{29} - 848300707458 z^{28} - 355077245616 z^{27} + \\ & 1794280909542 z^{26} - 467081765816 z^{25} - 922373507592 z^{24} + 312051187944 z^{23} - \\ & 1321527938829 z^{22} + 256058214800 z^{21} + 1870245875859 z^{20} + 455245169152 z^{19} - \\ & 685952191995 z^{18} - 410889238072 z^{17} - 463142659512 z^{16} - 436655545808 z^{15} + \\ & 1478403246906 z^{14} - 1591217568 z^{13} - 1100965032744 z^{12} - 137175061936 z^{11} - \\ & 672625854345 z^{10} + 130115107936 z^9 + 1183740872769 z^8 + 880615121736 z^7 - \\ & 470011423005 z^6 - 356422892616 z^5 + 207556342200 z^4 - 913159201752 z^3 + \\ & 369523810776 z^2 + 926067028376 z - 751138064892 \\ & -4505097105248 z^{137} - 544781681936 z^{138} \end{aligned} $
(43, 44, 45)		$ \begin{aligned} & -561535751331132078651839819235968256 z^{132} - \\ & 255900474003210216622468298452205568 z^{131} \end{aligned} $
49	149	$ \begin{aligned} & 15572671419924 z^{49} - 3772543101346 z^{48} - 16300018328191 z^{47} + 4319661615519 z^{46} - \\ & 4488346337109 z^{45} + 5940727404948 z^{44} + 17964463312288 z^{43} - 9518113636788 z^{42} - \\ & 23465636694092 z^{41} - 9034214618364 z^{40} + 13310977691102 z^{39} - 3432119444066 z^{38} + \\ & 4541780918154 z^{37} - 4658180508108 z^{36} - 15817903132432 z^{35} + 10273041871757 z^{34} + \\ & 16891806144813 z^{33} + 11029947581076 z^{32} - 8975515997286 z^{31} + 3990991996645 z^{30} + \\ & 1843162338871 z^{29} + 2819751726454 z^{28} + 5712528940514 z^{27} - 3123325650527 z^{26} - \\ & 4568857516124 z^{25} + 3079446451107 z^{24} - 1561054956031 z^{23} + 2253742987675 z^{22} + \\ & 8070075436642 z^{21} - 5285683683185 z^{20} - 6018866523615 z^{19} + 2242220741691 z^{18} - \\ & 16850715359180 z^{17} + 10042194978521 z^{16} + 21478221788849 z^{15} - \\ & 7030756589503 z^{14} + 2147040772453 z^{13} - 6644126415292 z^{12} - 20281107186208 z^{11} + \\ & 10615688297782 z^{10} + 23604136877588 z^9 + 4747990415974 z^8 - 9464569013826 z^7 + \\ & 554641056330 z^6 - 10417936355944 z^5 + 7521510211014 z^4 + 18118150311174 z^3 - \\ & 9798756899644 z^2 - 10300388851658 z - 14410340200754 \\ & -4599090143590 z^{50} + 47240971614 z^{49} - 7371501379626 z^{48} + 3684666222245 z^{47} - \\ & 2239828388971 z^{46} - 12147728070459 z^{45} + 1592748784080 z^{44} + 8564296897524 z^{43} - \\ & 965625646126 z^{42} + 10131713521668 z^{41} - 11133490379610 z^{40} - 15177903407224 z^{39} + \\ & 3577268952172 z^{38} - 2784273097486 z^{37} + 11204317248478 z^{36} + 14206959307062 z^{35} - \\ & 6344191607115 z^{34} - 2797061583877 z^{33} - 6009296669398 z^{32} - 9068120053960 z^{31} + \\ & 2840232781361 z^{30} + 11928343993591 z^{29} - 7364408409198 z^{28} - 1772349624642 z^{27} + \\ & 496840832203 z^{26} - 19016054531576 z^{25} + 9858371330193 z^{24} + 19295872493353 z^{23} - \\ & 4521827637741 z^{22} + 1208342237300 z^{21} - 4374640110055 z^{20} - 18502749486377 z^{19} + \\ & 6104435716535 z^{18} + 19146661474462 z^{17} + 488274966683 z^{16} - 3316326725319 z^{15} + \\ & 2182282334815 z^{14} - 14824597559769 z^{13} + 7532277927728 z^{12} + 16314787492980 z^{11} - \\ & 6917870993576 z^{10} - 723482062748 z^9 - 8442497375404 z^8 - 10800247171176 z^7 + \\ & 1817577563136 z^6 + 7697242742116 z^5 - 521297676820 z^4 + 2233470317936 z^3 - \\ & 420000649522 z^2 - 2445166706486 z - 554792510556 \\ & -44864565312812 z^{149} - 68768563833944 z^{150} \end{aligned} $

50	152	$70701440915972 z^{50} + 2485750409310 z^{49} - 48433419862414 z^{48} - 4768774365096 z^{47} -$ $43697106743944 z^{46} - 53486277061320 z^{45} + 52228629883448 z^{44} +$ $113082234644520 z^{43} - 54801050915832 z^{42} - 64681454517156 z^{41} +$ $173187609378876 z^{40} + 4321810781880 z^{39} + 38273938178088 z^{38} +$ $41634873091296 z^{37} - 48143733242184 z^{36} - 109412908047690 z^{35} +$ $9716456542658 z^{34} + 80250859917927 z^{33} - 168465758454760 z^{32} -$ $47271090077022 z^{31} - 10584097269228 z^{30} + 41168723771976 z^{29} -$ $21017776669584 z^{28} + 5162721672354 z^{27} + 7283096893976 z^{26} - 41509416153159 z^{25} -$ $13740013133908 z^{24} + 24267526979784 z^{23} + 2091821742234 z^{22} +$ $29437466574372 z^{21} + 9032398055592 z^{20} - 60386478472440 z^{19} -$ $73936925599220 z^{18} + 18728137584669 z^{17} + 3482309709856 z^{16} +$ $37500696189300 z^{15} + 47510069761656 z^{14} + 12055597311288 z^{13} -$ $35159704302024 z^{12} - 96663979023996 z^{11} + 40073524325696 z^{10} +$ $86053467392322 z^9 - 128842256441152 z^8 - 18762432745968 z^7 - 29965966283212 z^6 -$ $63882704291400 z^5 + 39994318420184 z^4 + 139982181869676 z^3 - 668440393388 z^2 -$ $78045602999628 z + 162004856572564$
		$36933555661356 z^{51} + 47920471863576 z^{50} - 55361681802540 z^{49} +$ $88101855745410 z^{48} + 75160967295972 z^{47} - 58185505771584 z^{46} -$ $89151046727544 z^{45} + 78638578340376 z^{44} + 28944982607844 z^{43} +$ $32365270828792 z^{42} + 70767237608154 z^{41} - 2897837630372 z^{40} -$ $101584076501208 z^{39} + 3986832279908 z^{38} + 909499929144 z^{37} - 47865768754168 z^{36} +$ $108831322045818 z^{35} + 6264960278182 z^{34} - 90486535189641 z^{33} +$ $62189759907232 z^{32} + 10409686675518 z^{31} + 31099124171420 z^{30} +$ $22971196235304 z^{29} + 35718305011712 z^{28} - 18763618774458 z^{27} -$ $83115522399416 z^{26} - 26374290250401 z^{25} + 22490348502472 z^{24} +$ $35347663844856 z^{23} + 15415934152918 z^{22} + 26457758570172 z^{21} -$ $45284574830120 z^{20} - 88493800774524 z^{19} + 72588911514944 z^{18} +$ $56207093663109 z^{17} - 66424921001812 z^{16} + 8638528221384 z^{15} -$ $92292496625888 z^{14} - 26271602491416 z^{13} + 34816523179000 z^{12} +$ $34907926208424 z^{11} + 43851949251232 z^{10} + 8179745438964 z^9 + 16540887976792 z^8 -$ $22321126797840 z^7 + 39555650921560 z^6 - 14655642670176 z^5 + 6978532909192 z^4 +$ $48835302448260 z^3 - 26285180837460 z^2 - 39059008911378 z + 32278938483072$
51	155	$10869781914776 z^{152} + 87172975855064 z^{153}$
		$13210080195340 z^{51} + 4522603679090 z^{50} - 9427546683654 z^{49} - 2774004593602 z^{48} -$ $7400333366216 z^{47} - 18389042628384 z^{46} + 12548156717848 z^{45} +$ $27341468684832 z^{44} - 15441422259600 z^{43} - 12543292885956 z^{42} +$ $36178289170772 z^{41} + 3532845840508 z^{40} + 6486818170568 z^{39} + 14736874206568 z^{38} -$ $11142318969776 z^{37} - 27268258119784 z^{36} + 6499641451100 z^{35} +$ $15552688098917 z^{34} - 35741550364067 z^{33} - 14297220262025 z^{32} +$ $320638622102 z^{31} + 6579237422834 z^{30} - 6364978145088 z^{29} + 4708521934772 z^{28} +$ $1245708648396 z^{27} - 10366430267285 z^{26} - 265982221073 z^{25} + 1459777281547 z^{24} -$ $1127570531968 z^{23} + 10634813564780 z^{22} + 287337494344 z^{21} - 12986467293314 z^{20} -$ $10770234249082 z^{19} - 3992766908299 z^{18} - 370829803149 z^{17} + 11556474774739 z^{16} +$ $6599951121868 z^{15} + 10041200526700 z^{14} - 6971238164168 z^{13} - 26523100115088 z^{12} +$ $11857021456152 z^{11} + 18519668298230 z^{10} - 29057767792850 z^9 - 2806926416618 z^8 -$ $3971521872128 z^7 - 21968919390312 z^6 + 10644380502624 z^5 + 33267432960812 z^4 -$ $6611034284340 z^3 - 11149342064328 z^2 + 34702276800324 z + 981145960816$

		$ \begin{aligned} & 9471847058336 z^{52} + 8059856303860 z^{51} - 12177152418460 z^{50} + 18749291515692 z^{49} + \\ & 21599773201460 z^{48} - 14444819794628 z^{47} - 20952257010196 z^{46} + \\ & 20136224631640 z^{45} + 1321024846664 z^{44} + 4077738215064 z^{43} + 24044002669190 z^{42} - \\ & 4263021234426 z^{41} - 20107393058282 z^{40} + 6339907974024 z^{39} - 9225054027280 z^{38} - \\ & 9725987385536 z^{37} + 28020194595724 z^{36} - 4139834380024 z^{35} - 14970556979527 z^{34} + \\ & 17056674980517 z^{33} - 4174676625749 z^{32} + 5502941089898 z^{31} + 7629557222846 z^{30} + \\ & 5037844502128 z^{29} + 681762605644 z^{28} - 14551651872212 z^{27} - 12917474596739 z^{26} + \\ & 5950916078361 z^{25} + 6715836234525 z^{24} + 390569499776 z^{23} + 12966422525876 z^{22} - \\ & 9308517476872 z^{21} - 24871693623182 z^{20} + 18287268771674 z^{19} + \\ & 10514655058465 z^{18} - 16446979062877 z^{17} + 7325484577311 z^{16} - \\ & 17826103849112 z^{15} - 10601837806464 z^{14} + 8519029342488 z^{13} + 6309601105656 z^{12} + \\ & 6465454287952 z^{11} + 7127843952492 z^{10} + 2525682365292 z^9 - 3622009476772 z^8 + \\ & 9053500583600 z^7 - 5281247431456 z^6 + 1470684239088 z^5 + 13831158561592 z^4 - \\ & 7428167087168 z^3 - 7434850976830 z^2 + 8602076610826 z - 3467479638186 \\ & - 62721998839472 z^{155} - 92277981264624 z^{156} \end{aligned} $
(49, 50, 51)		$ \begin{aligned} & 14680803386096826096942113490747540708160 z^{150} + \\ & 26623480618461132219167377379353660824672 z^{149} \end{aligned} $

Approximation polynomials and related determinants in Theorem 3

19	59	$ \begin{aligned} & -153 z^{19} - 4 z^{18} + z^{17} + 15 z^{13} + z^{11} + 56 z^{10} + 12 z^9 + 15 z^7 + z^5 + 56 z - 4 \\ & 153 z^{19} - 4 z^{18} - z^{17} - 15 z^{13} - z^{11} + 56 z^{10} - 12 z^9 - 15 z^7 - z^5 - 56 z - 4 \\ & 2 z^{59} + 30 z^{61} \end{aligned} $
20	62	$ \begin{aligned} & 41 z^{20} + z^{19} - 3 z^{18} - 4 z^{14} - 15 z^{11} - 3 z^{10} + z^9 - 4 z^8 - 15 z^2 + z + 1 \\ & -41 z^{20} + z^{19} + 3 z^{18} + 4 z^{14} - 15 z^{11} + 3 z^{10} + z^9 + 4 z^8 + 15 z^2 + z - 1 \\ & -8 z^{62} - 8 z^{64} \end{aligned} $
21	65	$ \begin{aligned} & 76 z^{21} + 2 z^{20} - 6 z^{19} - 2 z^{18} - 8 z^{15} - 28 z^{12} - 6 z^{11} + 2 z^{10} - 2 z^9 - 28 z^3 + 2 z^2 + 2 z - 2 \\ & -76 z^{21} + 2 z^{20} + 6 z^{19} - 2 z^{18} + 8 z^{15} - 28 z^{12} + 6 z^{11} + 2 z^{10} + 2 z^9 + 28 z^3 + 2 z^2 - 2 z - 2 \\ & -16 z^{65} - 16 z^{69} \end{aligned} $
(19, 20, 21)		$-8 z^{59}$
26	80	$ \begin{aligned} & -379 z^{26} + z^{25} + z^{24} + 11 z^{23} + z^{22} + z^{21} + 11 z^{20} + z^{19} + z^{18} + 141 z^{17} + z^{16} + z^{15} + \\ & 11 z^{14} + z^{13} + z^{12} + 11 z^{11} + z^{10} + z^9 + 141 z^8 + z^7 + z^6 + 11 z^5 + z^4 + z^3 + 11 z^2 + z + 1 \\ & 379 z^{26} + z^{25} - z^{24} + 11 z^{23} - z^{22} + z^{21} - 11 z^{20} + z^{19} - z^{18} + 141 z^{17} - z^{16} + z^{15} - \\ & 11 z^{14} + z^{13} - z^{12} + 11 z^{11} - z^{10} + z^9 - 141 z^8 + z^7 - z^6 + 11 z^5 - z^4 + z^3 - 11 z^2 + z - 1 \\ & -1040 z^{80} + 2 z^{82} \end{aligned} $
27	83	$ \begin{aligned} & 1798 z^{27} - z^{26} - z^{25} - 11 z^{24} - z^{23} - z^{22} - 11 z^{21} - z^{20} - z^{19} - 141 z^{18} - z^{17} - z^{16} - 11 z^{15} - \\ & z^{14} - z^{13} - 11 z^{12} - z^{11} - z^{10} - 141 z^9 - z^8 - z^7 - 11 z^6 - z^5 - z^4 - 11 z^3 - z^2 - z - 1419 \\ & -z^{26} + z^{25} - 11 z^{24} + z^{23} - z^{22} + 11 z^{21} - z^{20} + z^{19} - 141 z^{18} + z^{17} - z^{16} + 11 z^{15} - \\ & z^{14} + z^{13} - 11 z^{12} + z^{11} - z^{10} + 141 z^9 - z^8 + z^7 - 11 z^6 + z^5 - z^4 + 11 z^3 - z^2 + z + 379 \\ & -2 z^{83} - 2 z^{85} \end{aligned} $
28	108	$ \begin{aligned} & z^{28} + z^{27} - z - 1 \\ & z^{28} + z^{27} + z + 1 \\ & 2 z^{108} + 2 z^{109} \end{aligned} $
(26, 27, 28)		$1081600 z^{81} + 1081600 z^{80}$
39	119	$ \begin{aligned} & z^{39} + z^{38} - z^{12} - z^{11} \\ & z^{39} + z^{38} + z^{12} + z^{11} \\ & 2 z^{119} + 2 z^{120} \end{aligned} $
40	136	$ \begin{aligned} & -226980 z^{40} + 16380 z^{38} + 16380 z^{36} + 18720 z^{34} + 18720 z^{28} + 35100 z^{22} - 2340 z^{20} - \\ & 2340 z^{18} + 131040 z^{13} - 9360 z^{11} - 9360 z^9 - 9360 z^7 + 35100 z^4 - 2340 z^2 - 9360 z - 2340 \\ & 226980 z^{40} - 16380 z^{38} - 16380 z^{36} - 18720 z^{34} - 18720 z^{28} - 35100 z^{22} + 2340 z^{20} + \\ & 2340 z^{18} + 131040 z^{13} - 9360 z^{11} - 9360 z^9 - 9360 z^7 - 35100 z^4 + 2340 z^2 - 9360 z + 2340 \end{aligned} $

		$18720 z^{136} + 18720 z^{142}$
41	135	$-2340 z^{41} - 226980 z^{39} + 17836 z^{37} + 1552 z^{35} + 18832 z^{33} + 1552 z^{29} + 18832 z^{27} +$ $556 z^{23} + 35116 z^{21} - 2548 z^{19} + 1448 z^{14} + 131048 z^{12} - 10192 z^{10} - 776 z^8 - 9416 z^6 +$ $556 z^5 + 35116 z^3 - 776 z^2 - 2548 z - 9416$
		$2340 z^{41} + 226980 z^{39} - 17836 z^{37} - 1552 z^{35} - 18832 z^{33} - 1552 z^{29} - 18832 z^{27} -$ $556 z^{23} - 35116 z^{21} + 2548 z^{19} + 1448 z^{14} + 131048 z^{12} - 10192 z^{10} - 776 z^8 - 9416 z^6 -$ $556 z^5 - 35116 z^3 - 776 z^2 + 2548 z - 9416$
		$18832 z^{135} + 1552 z^{137}$
(39, 40, 41)		$88133760 z^{119} (1 + z)$

Approximation polynomials and related determinants in Theorem 4

17	53	$z^{17} + z^{16} - z^{15} + z^{14} - z^5 - z^4 - z^3 + z^2$ $z^{17} - z^{16} - z^{15} - z^{14} - z^5 + z^4 - z^3 - z^2$ $16 z^{53} - 16 z^{61}$
18	56	$2 z^{18} - 6 z^{17} - 5 z^{16} - z^{15} + 3 z^{14} + z^{13} - z^{12} + 3 z^{11} - 3 z^{10} - z^9 - z^8 + 3 z^7 + 3 z^6 +$ $z^5 + z^4 + 5 z^3 - 3 z^2 - 5 z - 8$
		$2 z^{18} + 6 z^{17} - 5 z^{16} + z^{15} + 3 z^{14} - z^{13} - z^{12} - 3 z^{11} - 3 z^{10} + z^9 - z^8 - 3 z^7 + 3 z^6 -$ $z^5 + z^4 - 5 z^3 - 3 z^2 + 5 z - 8$
		$-64 z^{56} - 64 z^{58}$
19	59	$z^{19} + z^{18} + z^{17} - z^{16} - z^{15} - z^{14} + z^{13} - z^{12} + z^{11} + z^{10} + z^9 - z^8 - z^7 - z^6 + z^5 -$ $z^4 + 2 z^3 + 2 z^2 + 2 z - 2$
		$z^{19} - z^{18} + z^{17} + z^{16} - z^{15} + z^{14} + z^{13} + z^{12} + z^{11} - z^{10} + z^9 + z^8 - z^7 + z^6 + z^5 +$ $z^4 + 2 z^3 - 2 z^2 + 2 z + 2$
		$32 z^{59} + 32 z^{67}$
(17, 18, 19)		$-512 z^{53}$
21	65	$2 z^{21} + 2 z^{20} - 2 z^{19} + 2 z^{18} - z^{17} - z^{16} - z^{15} + z^{14} - z^{13} - z^{12} - 3 z^{11} + 3 z^{10} - z^9 -$ $z^8 - 5 z^7 + 5 z^6 - z^5 - z^4 - 3 z^3 + 3 z^2 - 4 z - 4$
		$2 z^{21} - 2 z^{20} - 2 z^{19} - 2 z^{18} - z^{17} + z^{16} - z^{15} - z^{14} - z^{13} + z^{12} - 3 z^{11} - 3 z^{10} - z^9 +$ $z^8 - 5 z^7 - 5 z^6 - z^5 + z^4 - 3 z^3 - 3 z^2 - 4 z + 4$
		$-64 z^{65} - 64 z^{69}$
22	70	$4 z^{22} - 4 z^{21} + z^{20} + z^{19} + z^{18} - z^{17} - 2 z^{16} - 2 z^{15} + 4 z^{14} + 4 z^{13} + 6 z^{12} - 2 z^{11} -$ $4 z^{10} - 4 z^9 + 4 z^8 - 4 z^7 + 2 z^6 + 6 z^5 + 3 z^4 - 5 z^3 + 5 z^2 + 3 z - 8$
		$4 z^{22} + 4 z^{21} + z^{20} - z^{19} + z^{18} + z^{17} - 2 z^{16} + 2 z^{15} + 4 z^{14} - 4 z^{13} + 6 z^{12} + 2 z^{11} -$ $4 z^{10} + 4 z^9 + 4 z^8 + 4 z^7 + 2 z^6 - 6 z^5 + 3 z^4 + 5 z^3 + 5 z^2 - 3 z - 8$
		$128 z^{70} - 64 z^{72}$
23	71	$4 z^{23} - 4 z^{22} + z^{21} + z^{20} + z^{19} - z^{18} - 2 z^{17} - 2 z^{16} + z^{15} + z^{14} + 3 z^{13} + z^{12} - z^{11} -$ $z^{10} + z^9 - z^8 - z^7 + 3 z^6 - 2 z^4 + 2 z^3 - 5 z - 3$
		$4 z^{23} + 4 z^{22} + z^{21} - z^{20} + z^{19} + z^{18} - 2 z^{17} + 2 z^{16} + z^{15} - z^{14} + 3 z^{13} - z^{12} - z^{11} +$ $z^{10} + z^9 + z^8 - z^7 - 3 z^6 + 2 z^4 + 2 z^3 - 5 z + 3$
		$32 z^{71} - 64 z^{73}$
(21, 22, 23)		$3072 z^{65}$
26	80	$8 z^{26} + 5 z^{24} - 3 z^{23} + 5 z^{22} - 5 z^{21} - z^{20} - z^{19} + 3 z^{18} + z^{17} + 4 z^{16} - 4 z^{13} + 2 z^{12} -$ $2 z^{11} + 2 z^{10} + 2 z^9 + 3 z^8 - z^7 + 3 z^6 + z^5 - 5 z^4 - z^3 - 5 z^2 - 3 z - 8$
		$8 z^{26} + 5 z^{24} + 3 z^{23} + 5 z^{22} + 5 z^{21} - z^{20} + z^{19} + 3 z^{18} - z^{17} + 4 z^{16} + 4 z^{13} + 2 z^{12} +$ $2 z^{11} + 2 z^{10} - 2 z^9 + 3 z^8 + z^7 + 3 z^6 - z^5 - 5 z^4 + z^3 - 5 z^2 + 3 z - 8$
		$64 z^{80} + 64 z^{88}$
27	91	$4 z^{27} + 4 z^{26} + 4 z^{25} - 4 z^{24} + 2 z^{19} + 2 z^{18} + 2 z^{17} - 2 z^{16} - 2 z^{15} - 2 z^{14} + 2 z^{13} -$ $2 z^{12} + 2 z^{11} + 2 z^{10} + 2 z^9 - 2 z^8 + 2 z^7 + 2 z^6 - 2 z^5 + 2 z^4 - 4 z^3 - 4 z^2 - 4 z + 4$

		$4z^{27} - 4z^{26} + 4z^{25} + 4z^{24} + 2z^{19} - 2z^{18} + 2z^{17} + 2z^{16} - 2z^{15} + 2z^{14} + 2z^{13} + 2z^{12} + 2z^{11} - 2z^{10} + 2z^9 + 2z^8 + 2z^7 - 2z^6 - 2z^5 - 2z^4 - 4z^3 + 4z^2 - 4z - 4$
		$128z^{91} + 64z^{99}$
28	88	$2z^{28} + 4z^{27} + 8z^{26} + 4z^{25} + 2z^{24} - 4z^{23} + 4z^{22} - 4z^{21} + z^{19} + 3z^{18} + 3z^{17} + 2z^{16} - 2z^{15} - 2z^{14} - 2z^{13} - 2z^{11} + 4z^{10} + 2z^9 + 2z^8 + 4z^6 - 2z^4 - z^3 - 9z^2 - 3z - 6$
		$2z^{28} + 4z^{26} + 6z^{24} + 4z^{23} + 4z^{22} + 4z^{21} + z^{19} + z^{18} - z^{17} + 4z^{16} + 4z^{13} + 2z^{12} + 4z^{11} + 2z^{10} + 4z^8 + 2z^7 + 2z^6 - 2z^5 - 4z^4 - 3z^3 - 5z^2 - z - 10$
		$64z^{88} + 128z^{90}$
(26, 27, 28)		$-4096z^{80}$

Approximation polynomials and related determinants in Theorem 5

10	31	$z^{10} - z^9 - z^8 + z^6 - z^5 - 2z^4 + z^2 - z$
		$z^9 + 2z^8 - z^7 + 3z^5 + 2z^4 + 1$
		$z^{31} - z^{47}$
11	43	$z^9 + z^5 + z^4 + z + 1$
		$-z^{10} - z^9 - z^8 - z^6 - 2z^5 - z^4 - z^3 - z^2 - 2z - 2$
		$z^{43} - z^{64}$
12	43	$z^{10} + 3z^9 + z^6 + z^5 + z^4 + z^2 + z + 1$
		$-z^{11} - 4z^{10} - 4z^9 - 3z^8 - z^7 - 2z^6 - 2z^5 + z^4 - 2z^3 - 2z^2 - 3z - 4$
		$z^{43} - 2z^{44}$
(10, 11, 12)		$-3z^{32} - 2z^{31}$
26	79	$z^{24} + z^{20} + z^{16}$
		$-z^{25} - z^{24} - z^{23} - z^{21} - z^{20} - z^{17} - z^{16} - z^{15}$
		$-z^{79} - z^{207}$
27	83	$-z^{24} - z^{20} + z^4 + 1$
		$z^{25} + z^{24} + z^{21} + z^{20} + z^{19} - z^5 - z^4 - z^3 - z - 1$
		$z^{83} - z^{91}$
28	85	$z^{28} - z^{26} - 2z^{22} - z^{21} - 2z^{10} - 2z^9 - 2z^6 - 3z^5 + z^4 - z^2 - 2z + 1$
		$z^{26} + 2z^{23} + 3z^{22} + 2z^{21} + 2z^{11} + 4z^{10} + 4z^9 + 2z^8 + 2z^7 + 5z^6 + 2z^5 + 3z^2 + 2z$
		$z^{85} - z^{88}$
(26, 27, 28)		$-z^{80} - z^{79}$